Theoretical Computer Science exam 4BIM

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1 Exercise 1

- Let $G = \langle V, E \rangle$ be a bipartite graph.
  - Let i denote the number of nodes in its first set, and let $|V| - i$ be the number of nodes in its second one.
  - Compute $|E|$, assuming that G is s.t. every node of the first set is connected to every node of the second one.
  - Compute the value i that maximizes $|E|$ (explain every step).
- Consider the graph represented in Figure 1.
  - Compute the clustering coefficients of nodes a and b
  - Compute the average distance between node a and the remaining nodes (same for node b).

2 Exercise 2

This exercise aims at implementing (simplified) edge-reinforced random walks (ERRWs) on graphs. ERRWs have been used in [1], to model how concepts (ideas) are explored, and how innovations (new ideas) appear.

2.0.1 Graph

Let $G = \langle V, E, w \rangle$, be an weighted undirected graph. V and E are respectively the set of nodes, and the set of edges. Each node $v_i \in V$ represents a concept (an idea), and an edge $\langle v_i, v_j \rangle \in E$ exists if there is a relation between concepts v_i and v_j. The weight of an edge represent the strength of the relation between the two linked concepts. Let N and K denote respectively the number of nodes and the number of edges. Let us assume that the topology of the network, as well as N and K are fixed beforehand, while weights w will change along time.

- Use the networkx `connected_watts_strogatz_graph` function to create a Watts Strogatz random graph G. This function takes the number of

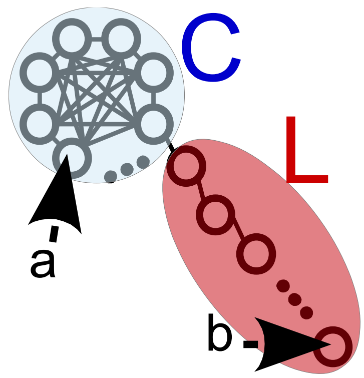

Figure 1: This graph is constituted by a clique with size C, and a chain with size L, both connected by a simple bridge.

nodes n (here $n = N = 100$), the average degree of each node k, and the rewiring probability p (here $p = 0.1$). Compute the value of k that allows to have $K = 500$ edges.

- Assign a weight $w_{ij} = 1$, $\forall \langle i, j \rangle \in E$.
- Plot the graph.

2.0.2 Walker

A random walker is an agent that jumps from one node to another adjacent node randomly, at each time step. The probability of jumping from node i to node j is proportional to the weight of the edge between both nodes: $Prob(i \rightarrow j) = \frac{w_{ij}}{\sum_l w_{il}}$. When the walker walks through an edge $\langle i, j \rangle \in E$, then the weight w_{ij} is increased: $w_{ij} \leftarrow w_{ij} + \delta_w$ (here $\delta_w = 1$).

- The walker is simply modeled by the node $v \in V$ where it is currently located. Initialize the location of the walker randomly (hint: use the `np.random.choice` function).
- Write a function that receives as inputs: the graph, and the current location of the walker; and outputs the new location, according to the previous probability (hint: use the `np.random.choice` function).
- Write a function that takes as inputs: the graph, the previous node of the agent and its current node; and updates the edge weight accordingly.
- Write a function that runs a random walk step and the weight update for a given number of iterations, (the number of iteration is a parameter of the function, as well as the graph).
- Run the previous function for 1000 iterations.

- Represent the distribution of the edge's weights using a histogram.

3 Exercise 3

We have seen in class the Max-CLIQUE Problem (given a graph $G = (V, E)$ and an integer k, find if G has a clique of at least k nodes) and proved it is NP-complete. Now we define a version where we consider only graphs in which the degree of every node is at most 4. Let's call this problem Max-Degree-4-Clique. More formally, given a graph $G = (V, E)$, such that $\forall v \in V \ d(v) \leq 4$ and given an integer k, find if G has a clique of at least k nodes.

(a) Prove that Max-Degree-4-Clique is in NP.

(b) What is wrong with the following proof of NP-completeness for Max-Degree-4-Clique? We know that the Max-CLIQUE problem in general graphs is NP-complete, so it is enough to present a reduction from Max-Degree-4-Clique to Max-CLIQUE. Given a graph G with nodes of degree at most 4, and an integer k, the reduction leaves the graph and the parameter k unchanged. Thus, $G' = G$ and $k' = k$ is clearly a possible input for the Max-CLIQUE problem. Furthermore, the answer to both problems is identical. This proves the correctness of the reduction and, therefore, the NP-completeness of Max-Degree-4-Clique.

(c) Describe a polynomial time algorithm for Max-Degree-4-Clique problem. What is the complexity of your algorithm?

4 Exercise 4

Consider the following decision problem: Given an undirected graph $G = (V, E)$ and an integer k, find if it is possible to color the nodes of G with two colors *red* and *blue* such that *both* of the followings hold:

(1) A node colored red is adjacent *only* to nodes colored blue.

(2) There are at least k nodes colored red.

Is this problem in P or NP-complete? If you think the problem is in P then write the pseudocode of a polynomial algorithm to solve it. Otherwise, provide the reduction for the NP-completeness.

References

[1] Iacopo Iacopini, Staša Milojević, and Vito Latora. Network dynamics of innovation processes. *Physical review letters*, 120(4):048301, 2018.