Graph Theory

Graph Traversal Algorithms

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Breadth-First Search

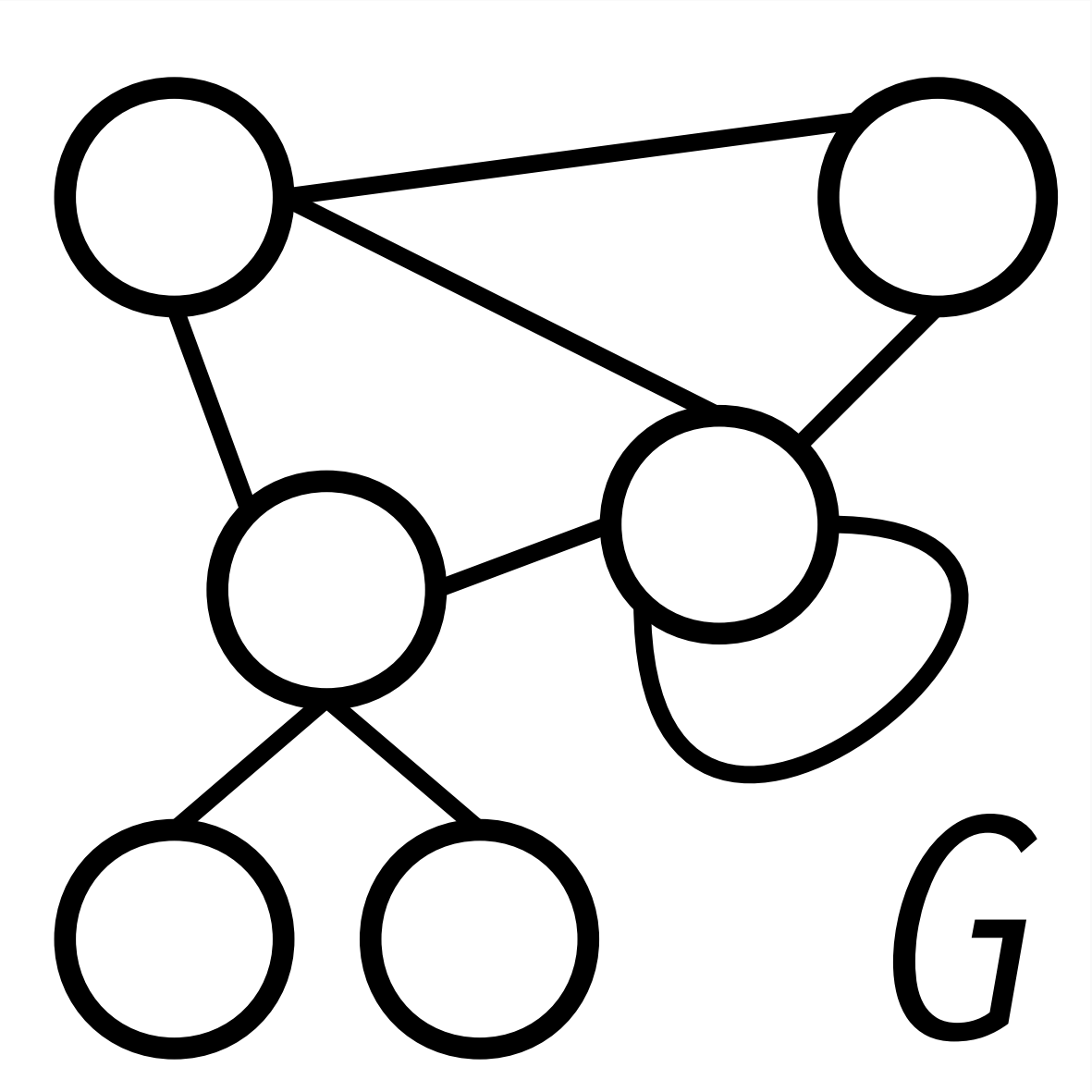
Breadth-First Search

- Algorithm to **traverse/search** in **graphs**.
- **Starts** at some **arbitrary node** *s*.
- **Visit all neighboring** nodes at the **present depth**.
- **Move** to nodes from the **next depth level**.

Pseudocode

```
function BFS(G = ⟨V, E⟩, s ∈ V)
    for all v ∈ V do
        state[v] ← undiscovered,    parent[v] ← ∅
    end for
    state[s] ← discovered,    parent[s] ← root
    Q ← [s] // Queue containing s
    while Size(Q) > 0 do
        u ← Q[0]
        for all v ∈ Neighbors(u) do
            if state[v] = undiscovered then
                state[v] ← discovered,    parent[v] ← u
                Q ← Q + [v] // Append new node
            end if
        end for
        Q ← Q[1 : Size(Q)] //Remove 1st node (FIFO)
    end while
    Return parent
end function
```

Example



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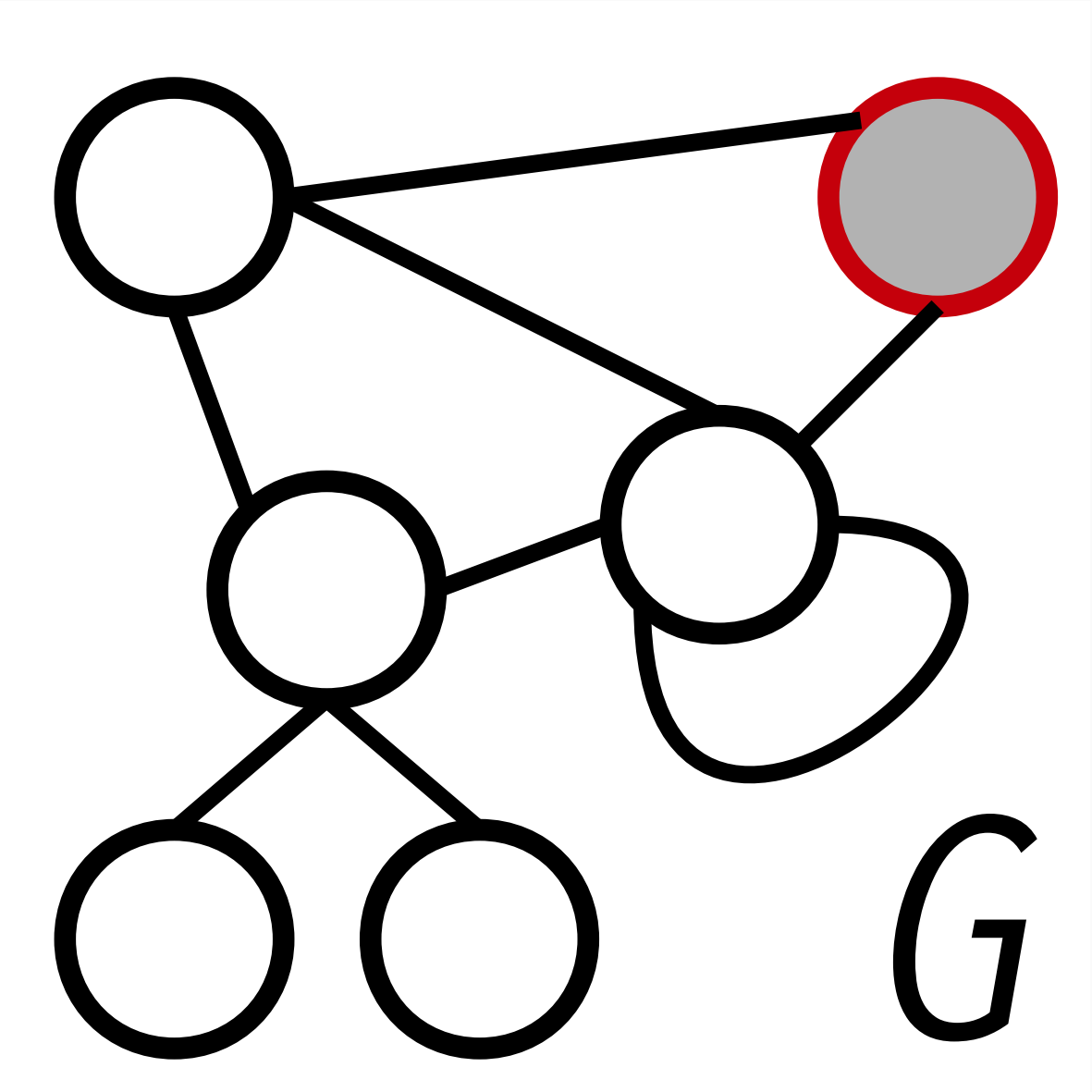

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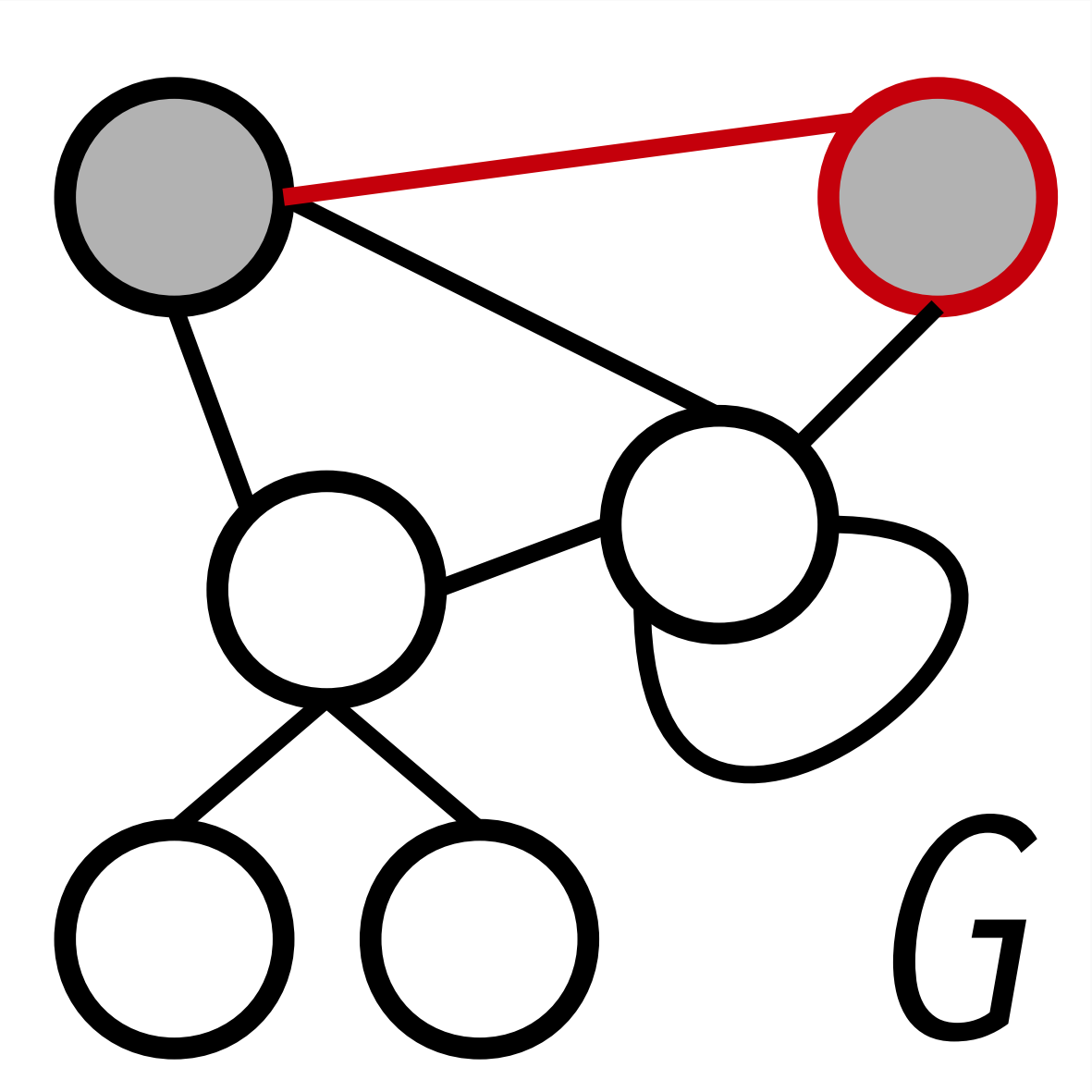

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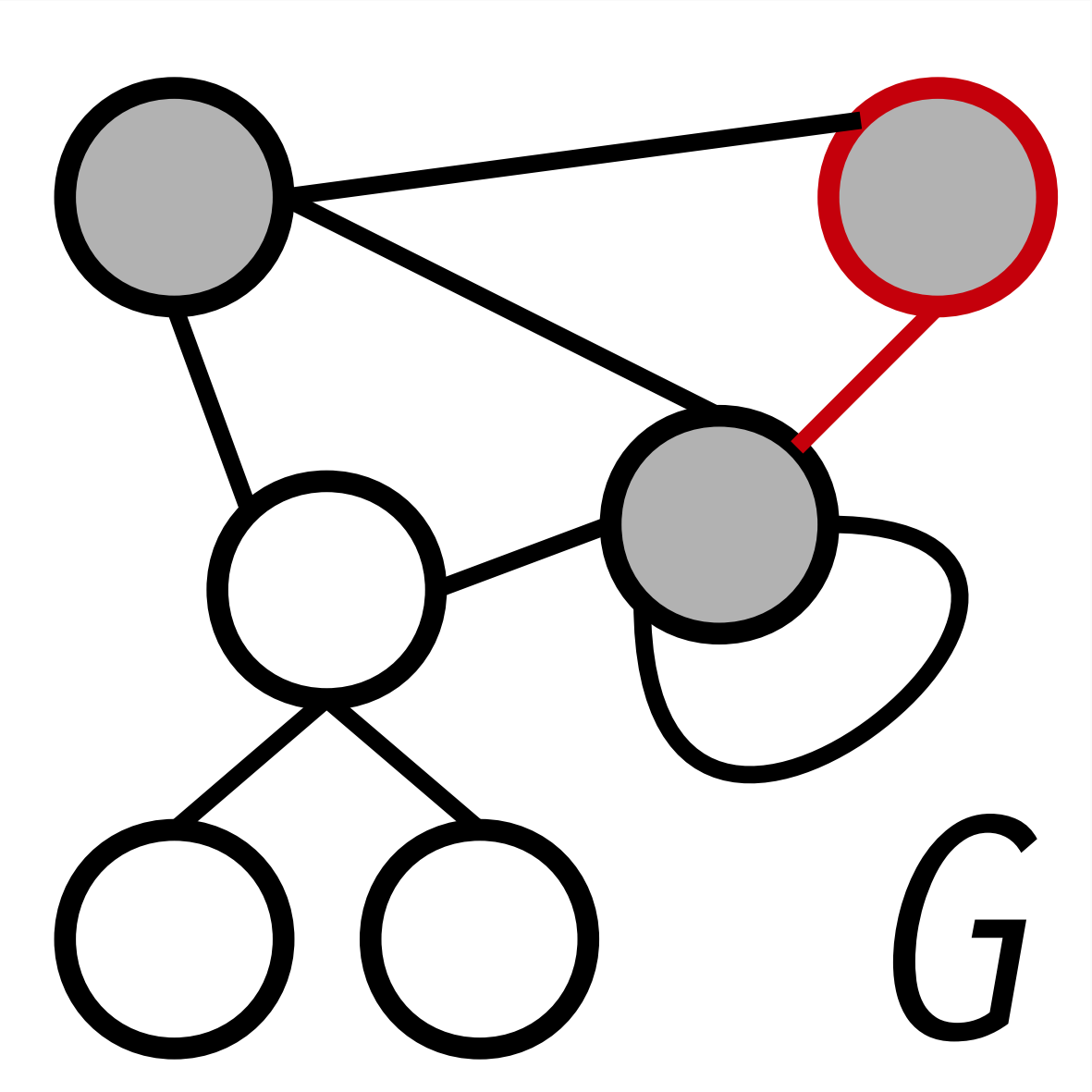

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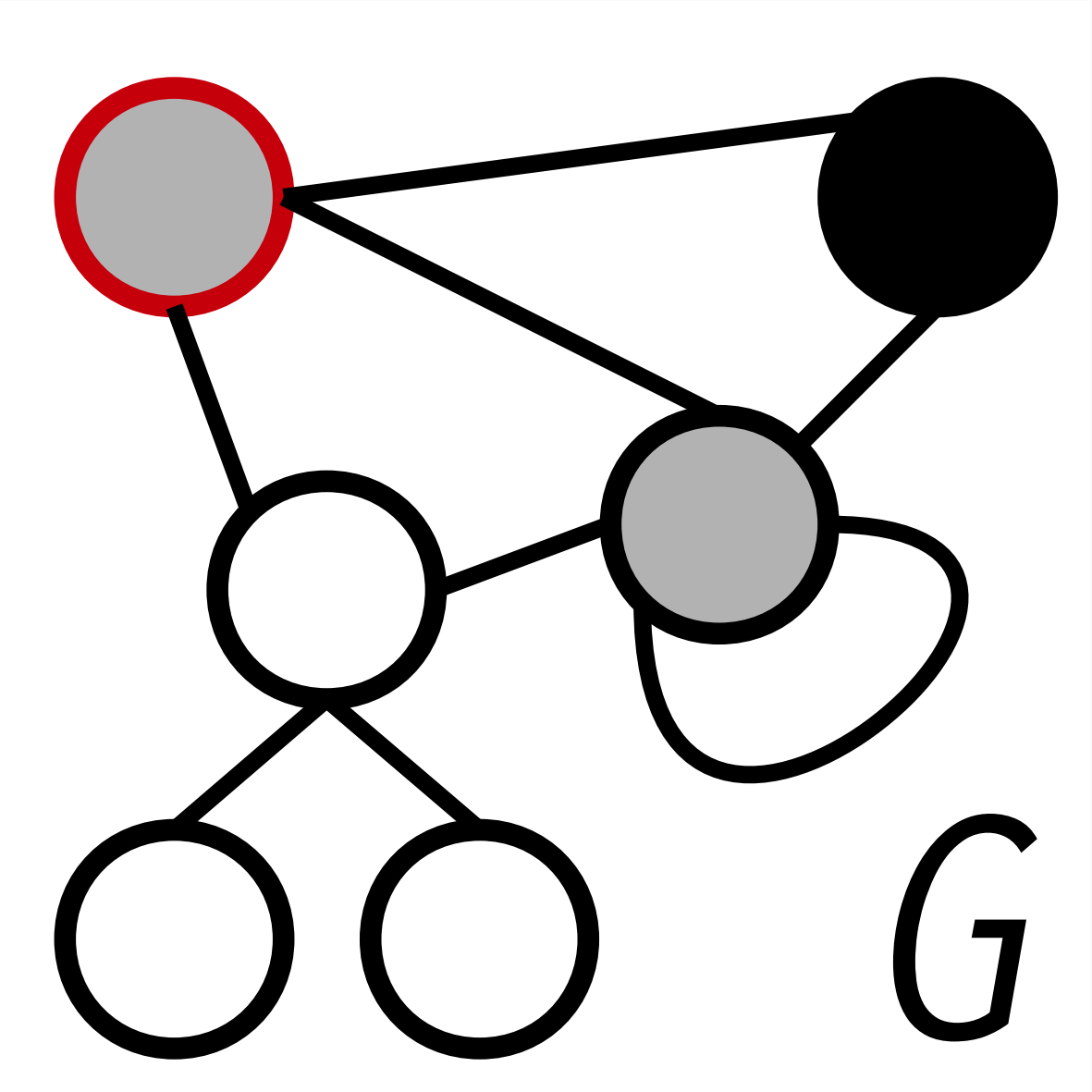

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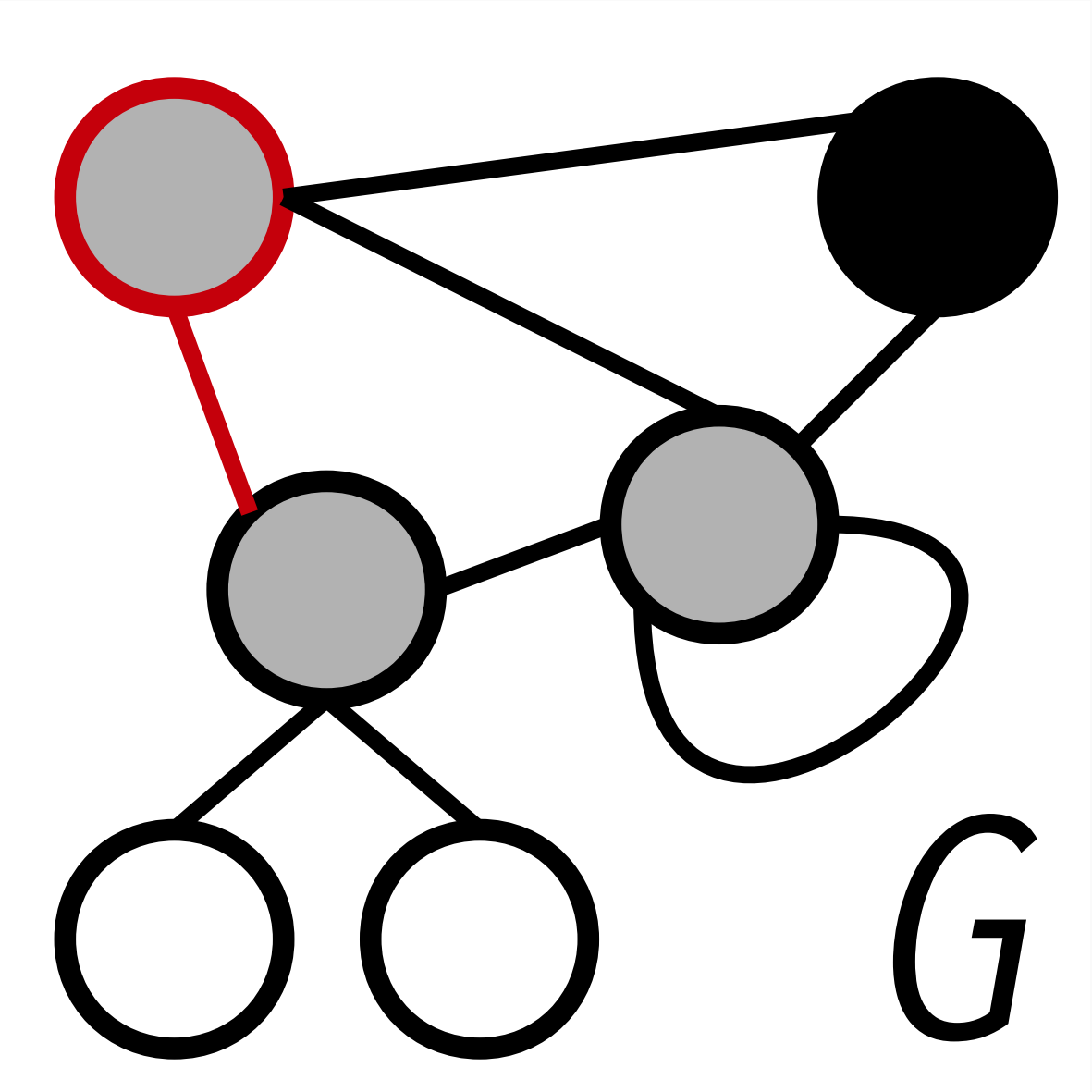

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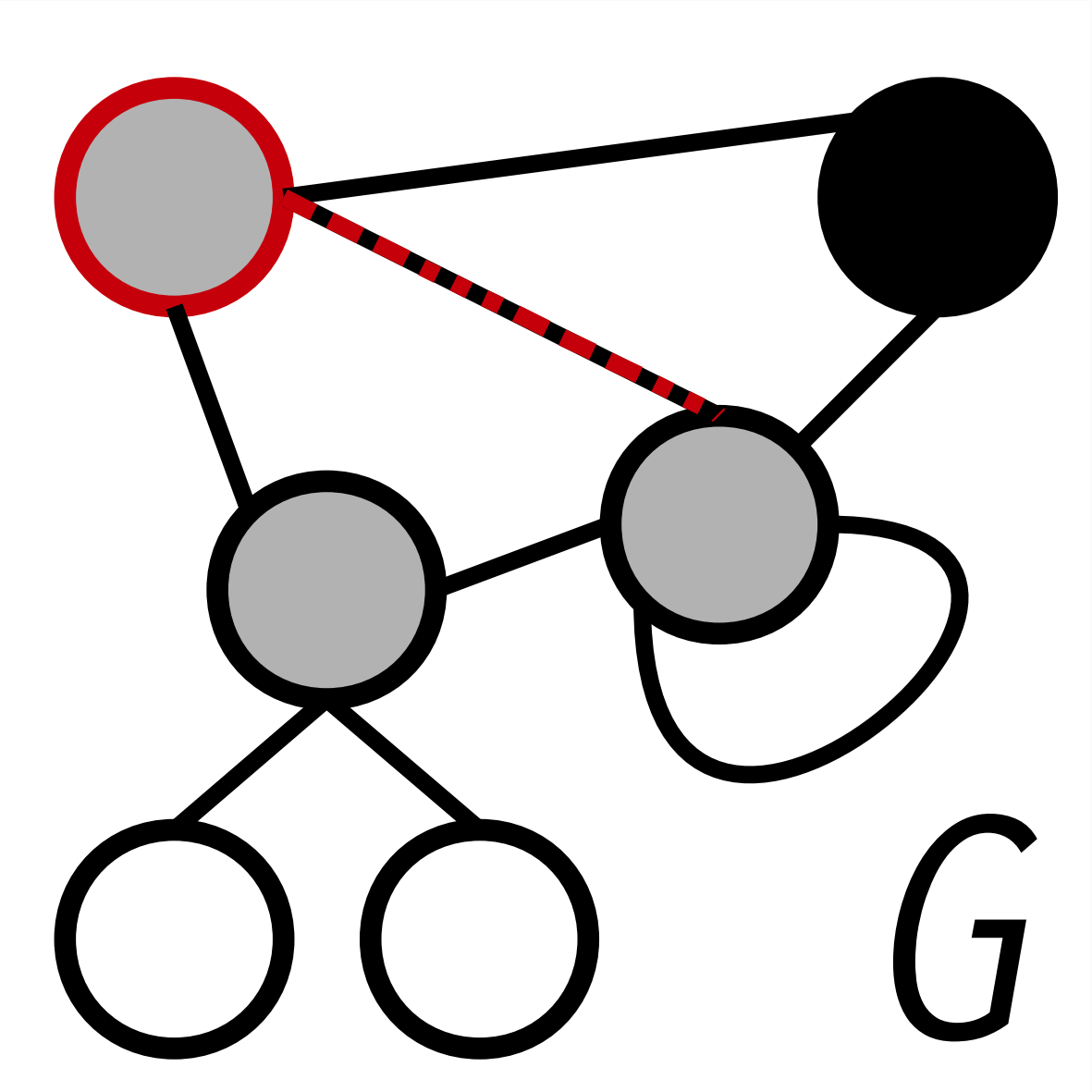

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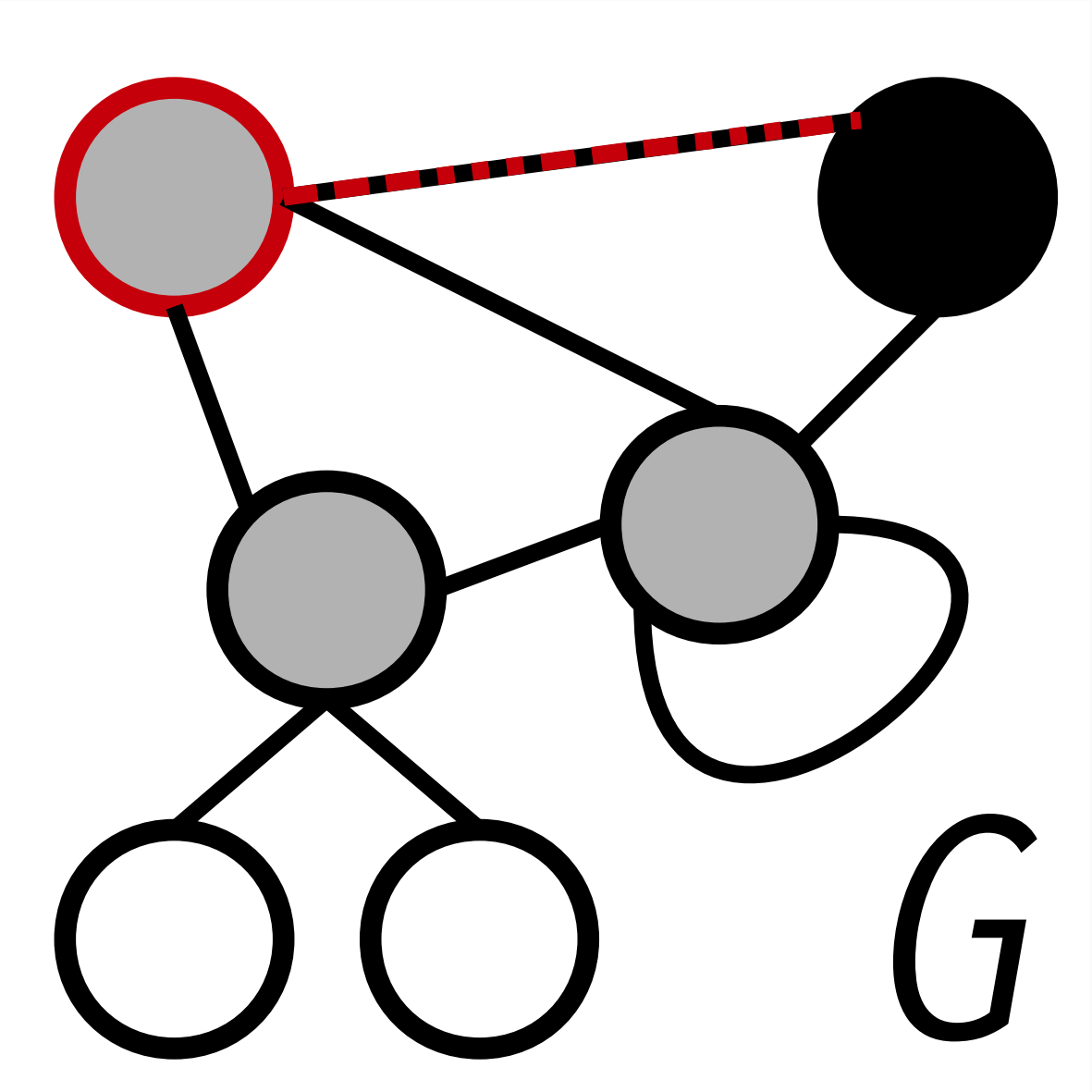

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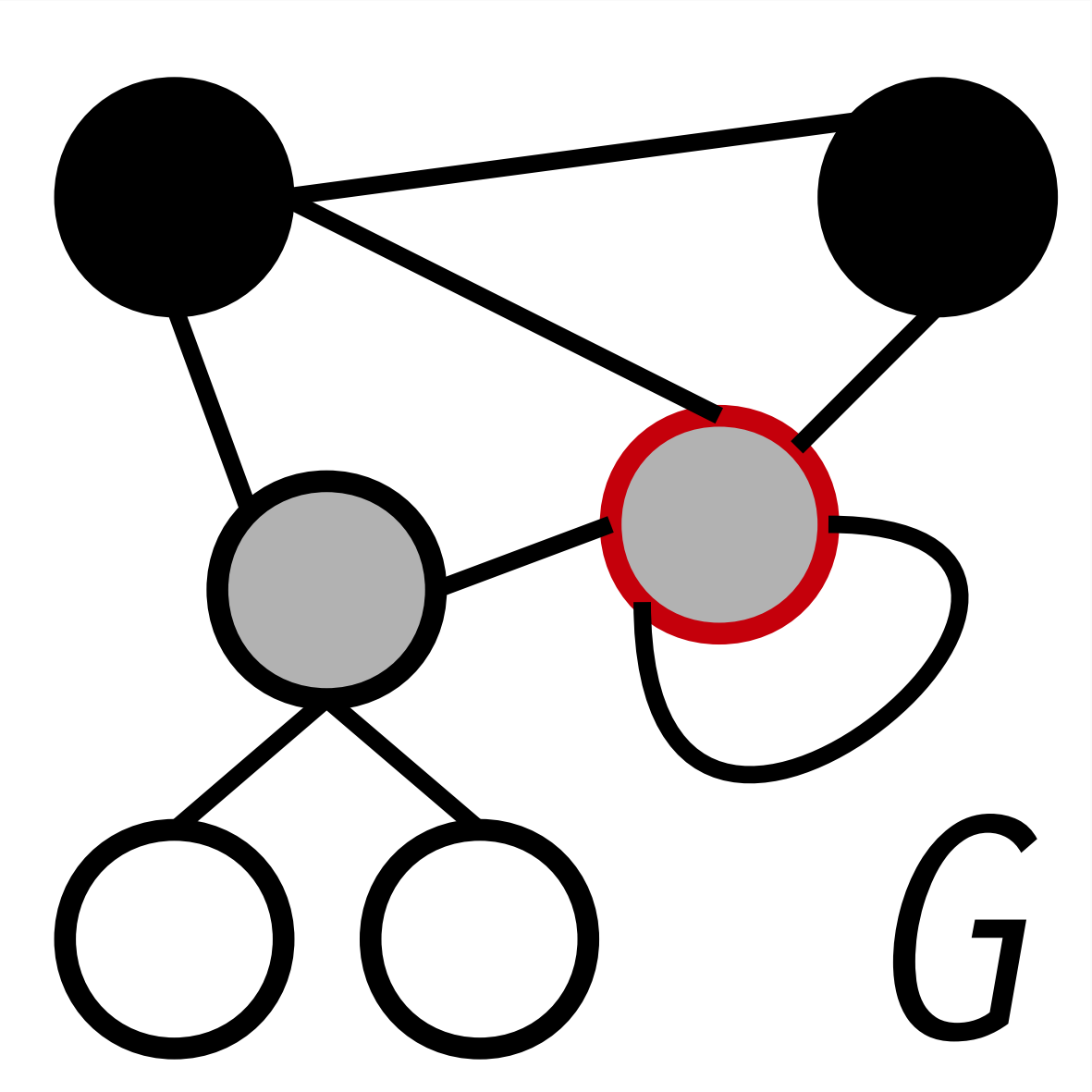

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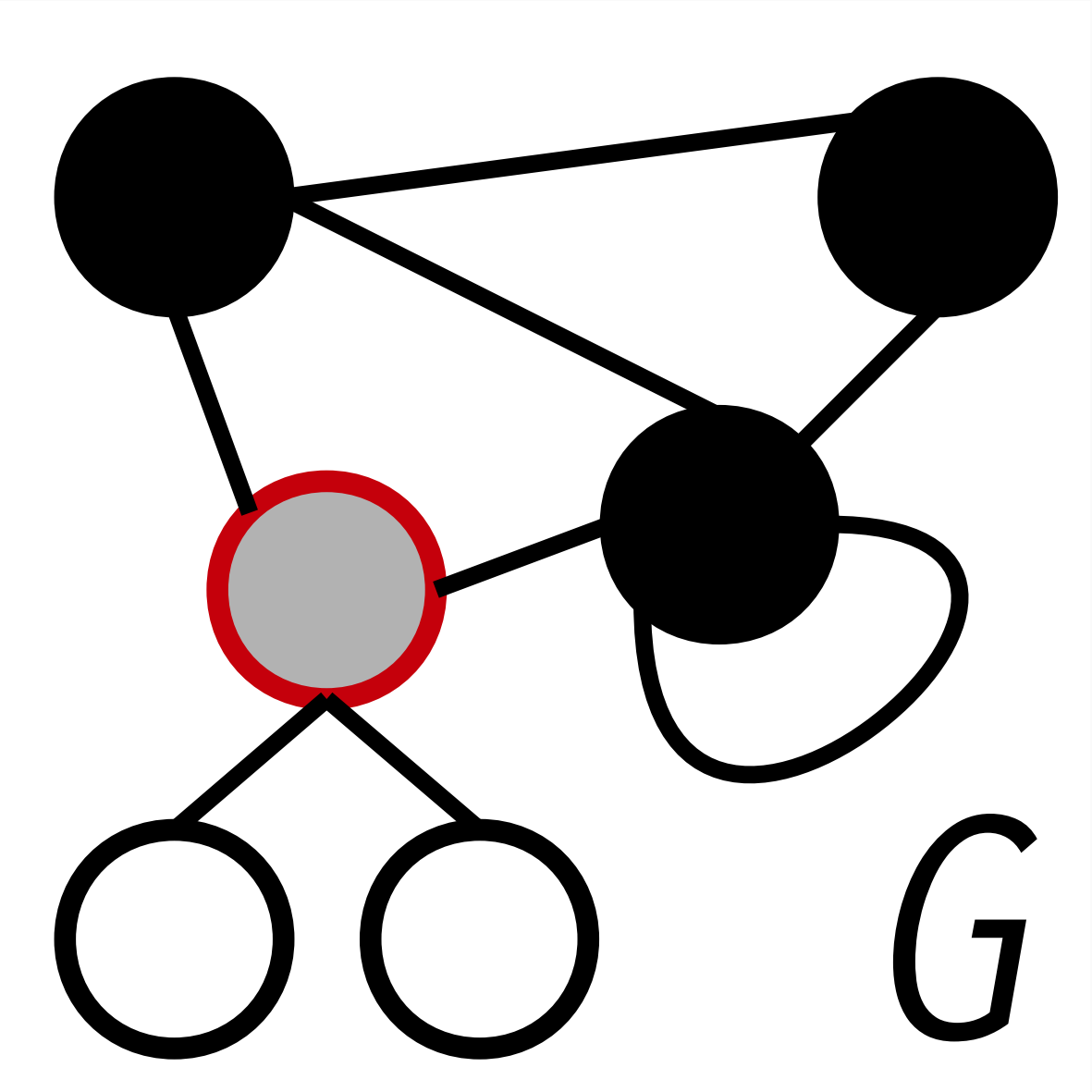

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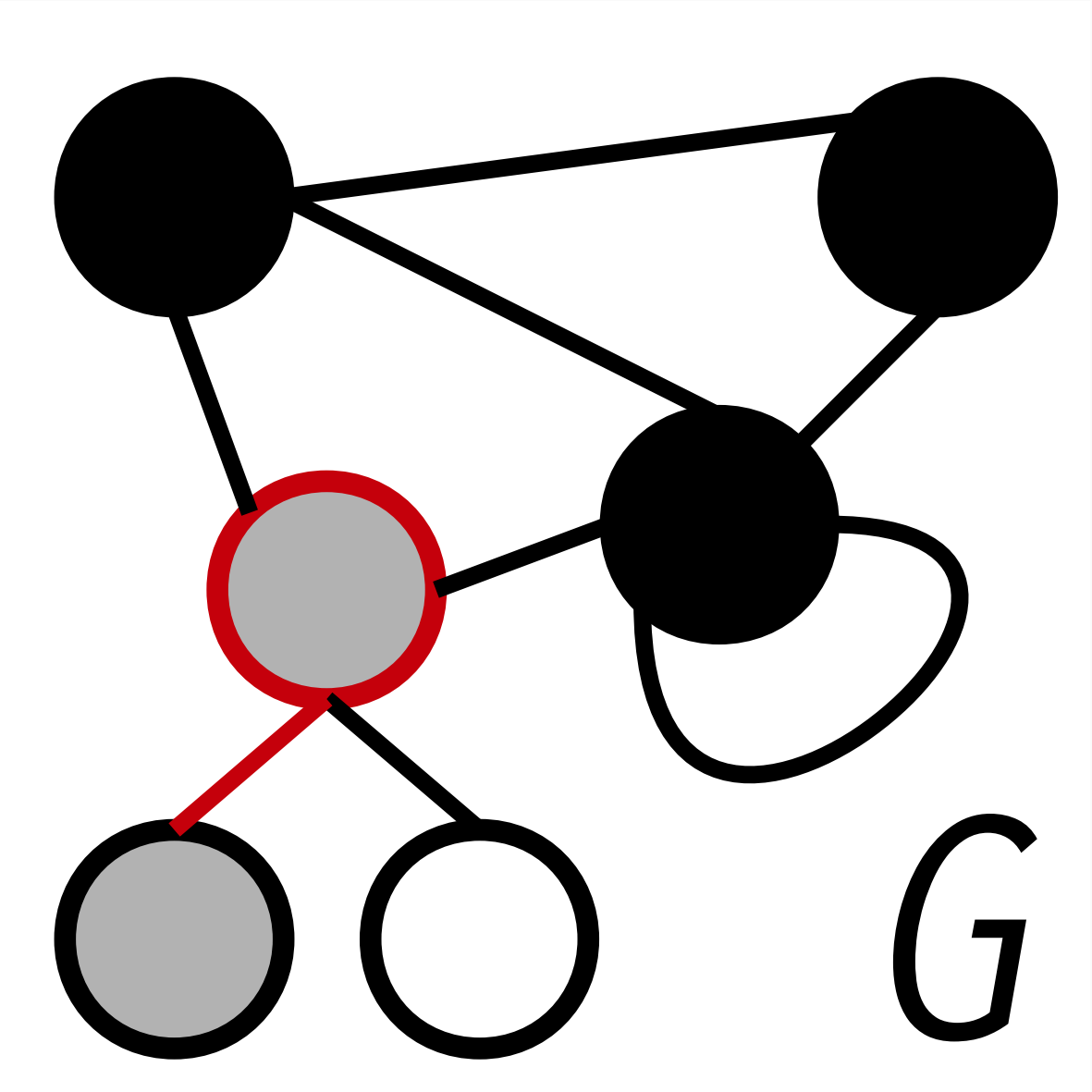

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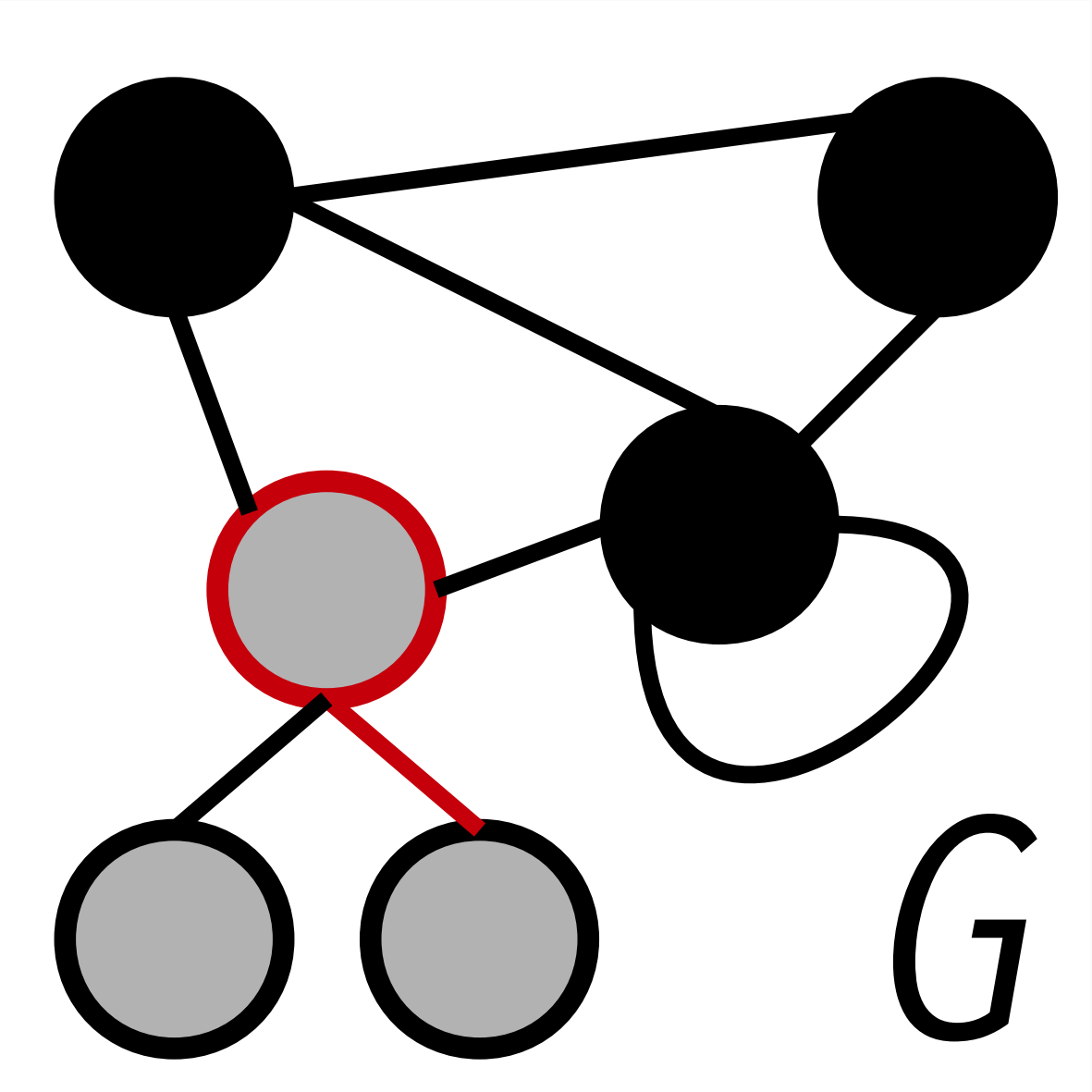

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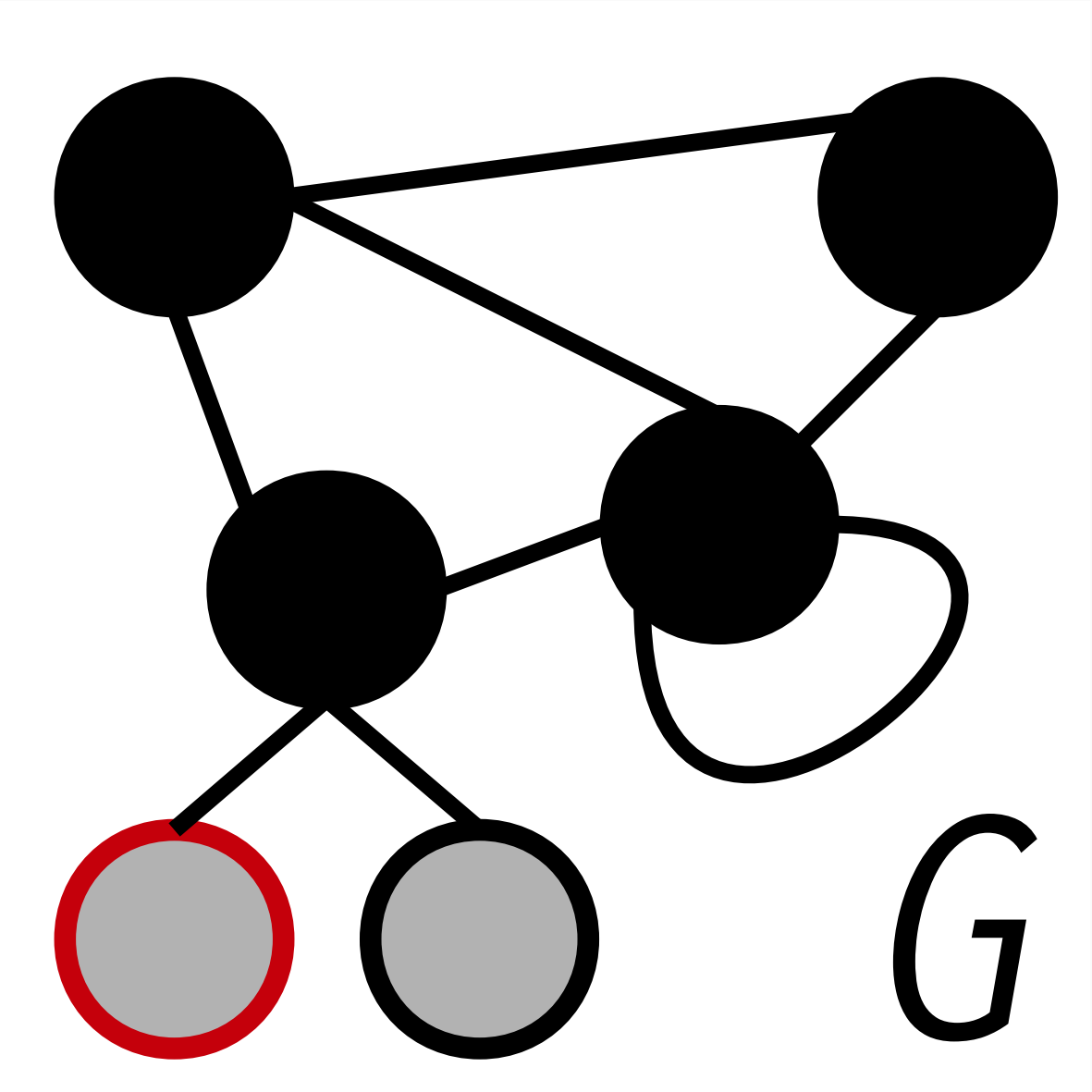

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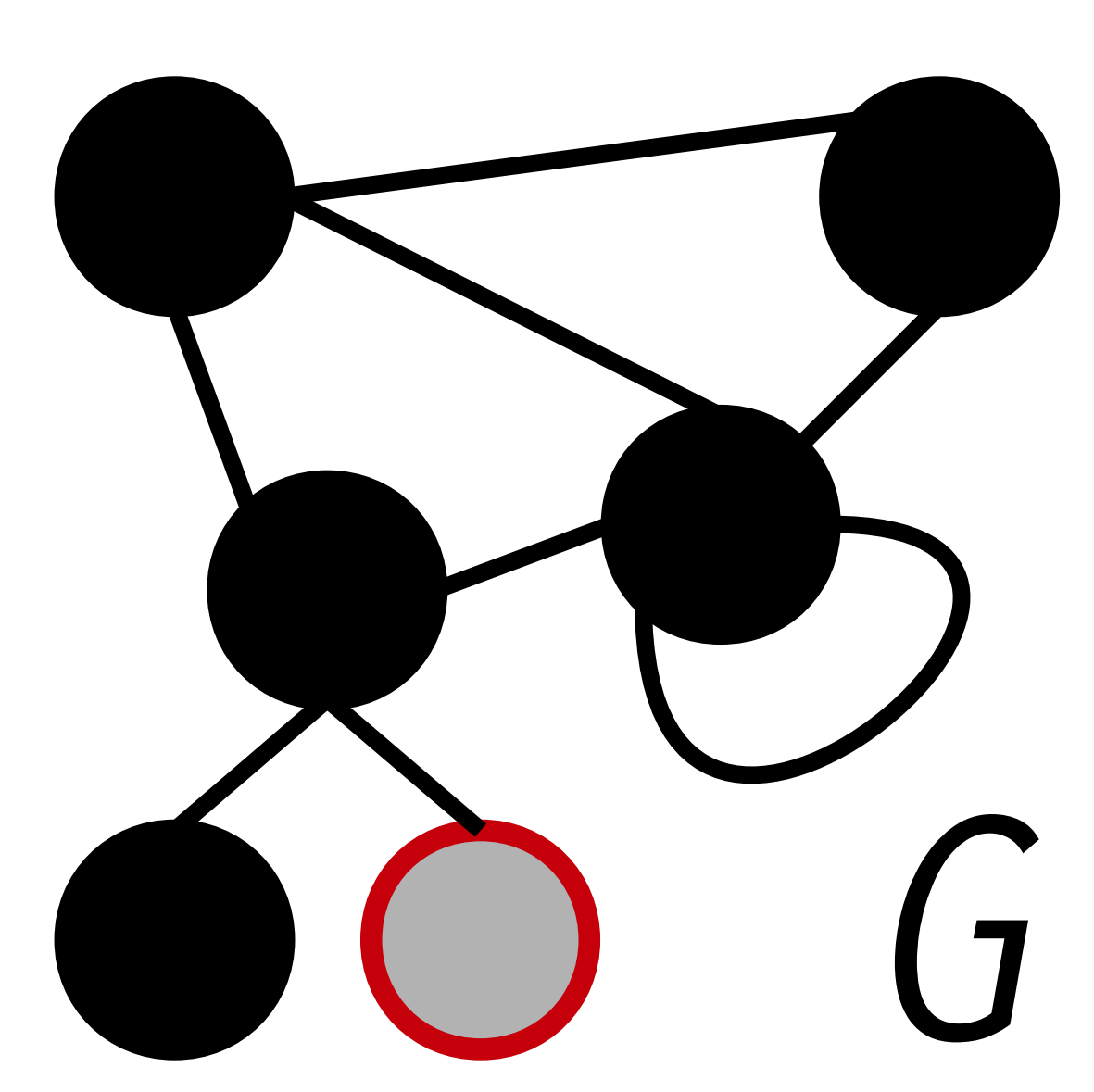

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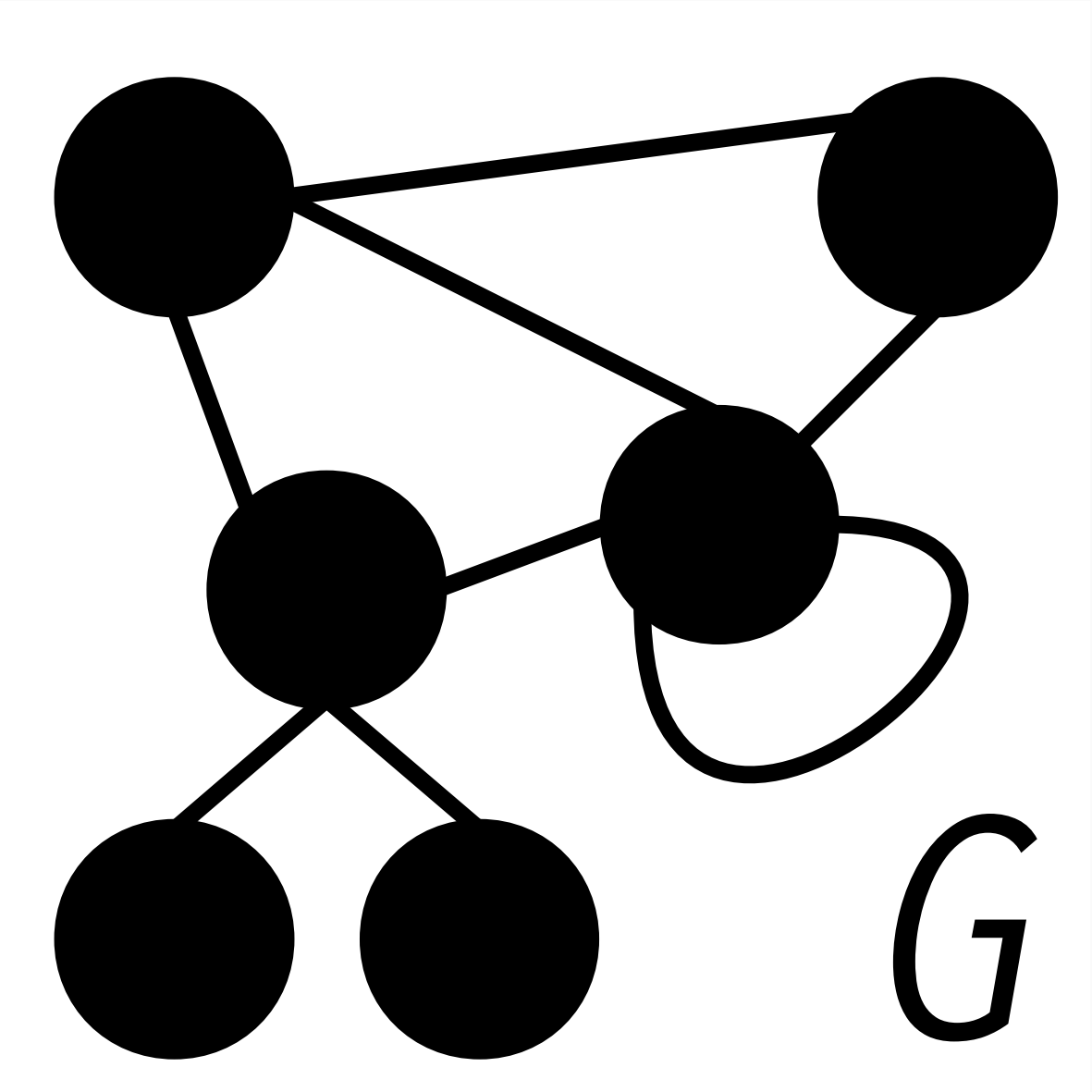

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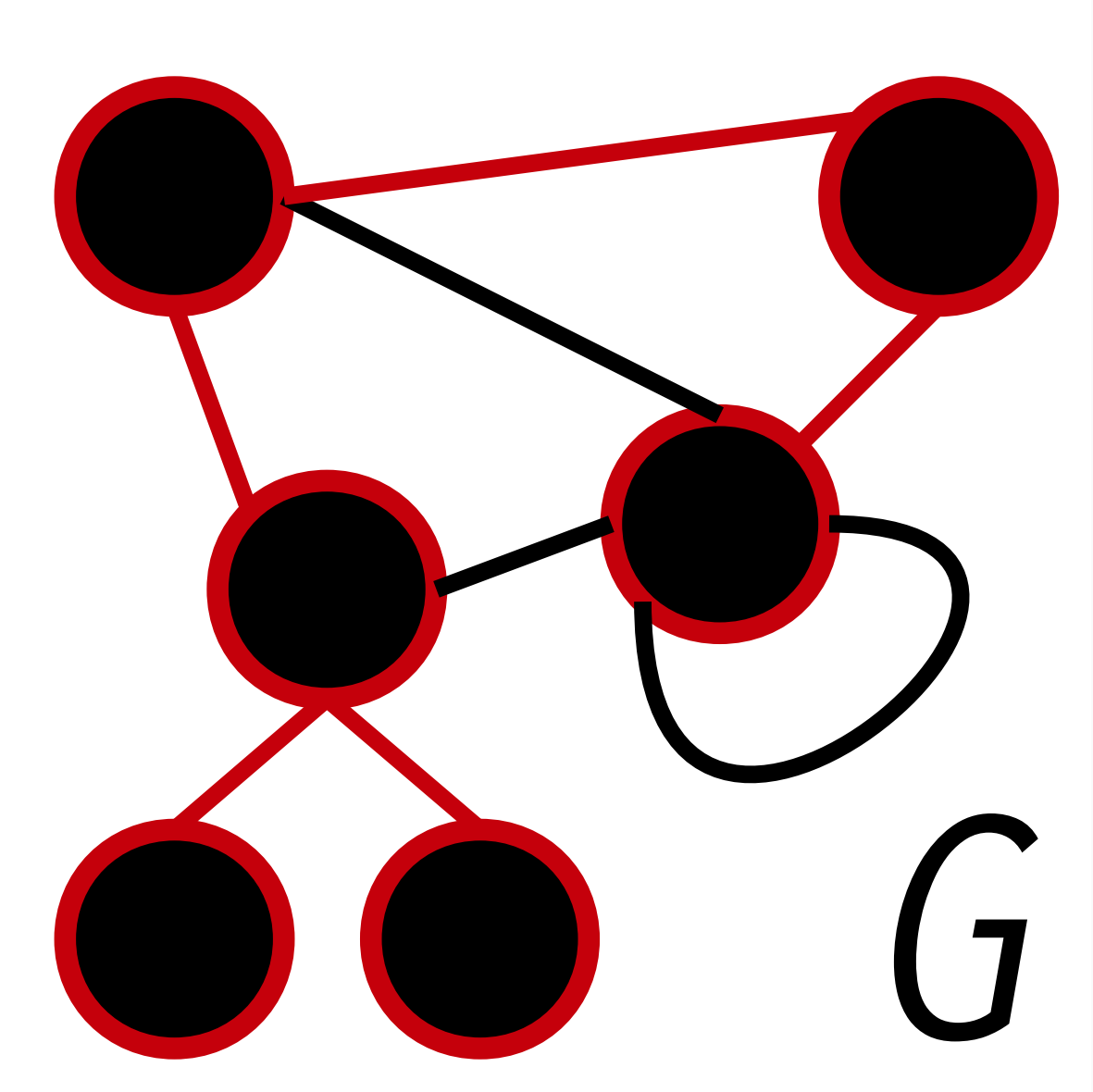


Analysis

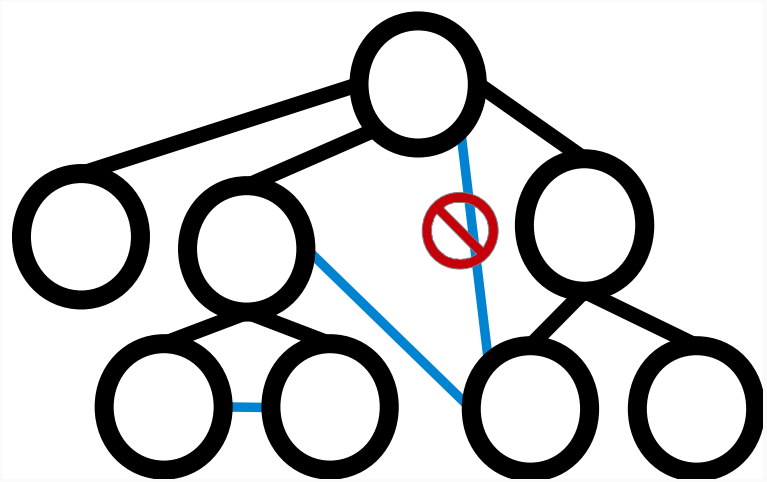
Remark: Memorize the nodes' levels using a counter.

Time Complexity: $\mathcal{O}(|E| + |V|)$ (Why?)

Property 1: BFS $\rightarrow$ spanning tree $T = \langle V, E' \rangle$ (pseudocode?).

Property 2: BFS $\rightarrow$ shortest path from s to $v \in V$ (Proof?).

Property 3: e : edge s.t. $e \in E$ and $e \notin E'$; e connects 2 nodes that are at most 1 level apart (Proof?).



Exercice

Modify the BFS Algorithm to check if G is a bipartite.

Depth-First Search

Depth-First Search

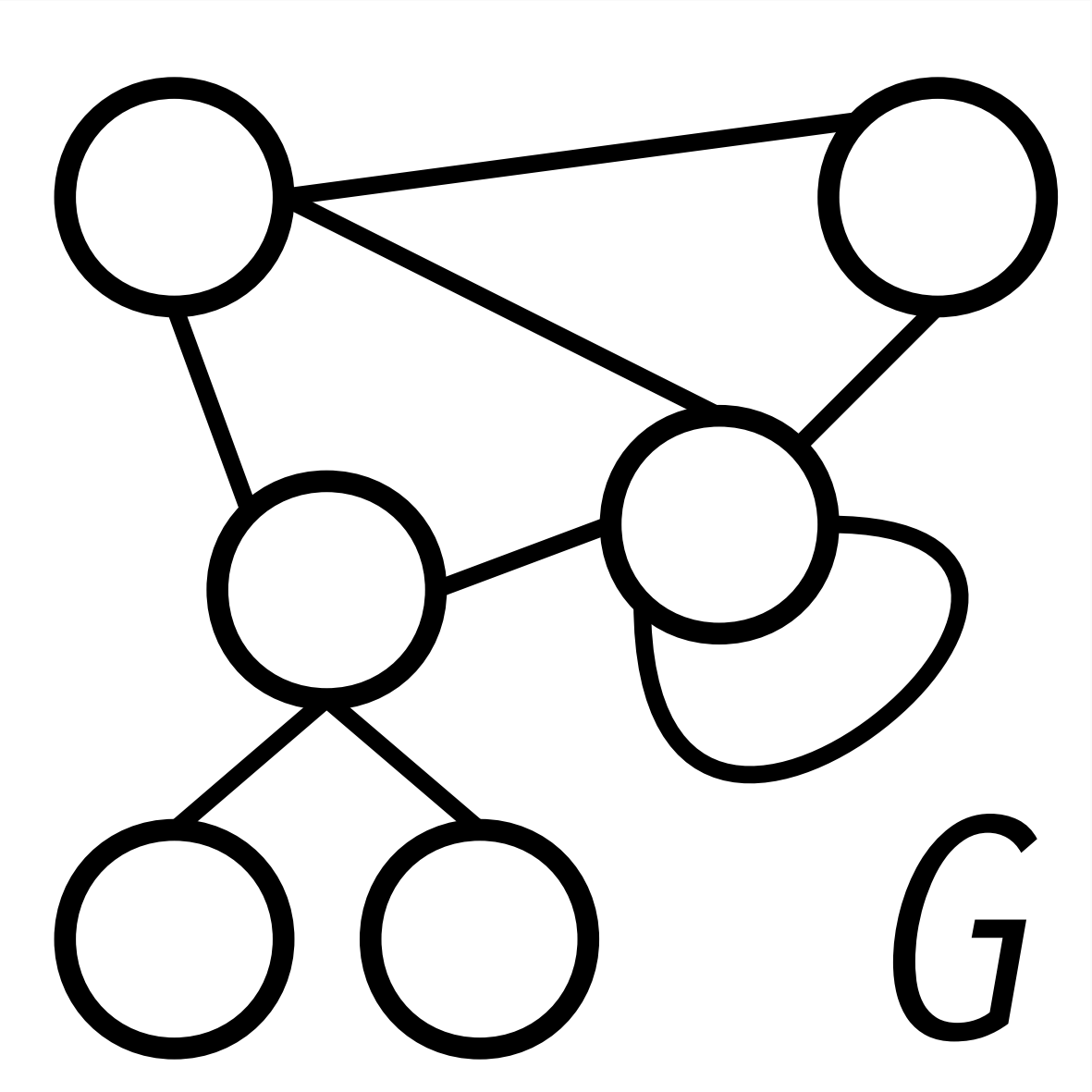
- Algorithm to **traverse/search** in **graphs**.
- **Starts** at some **arbitrary node** s.
- Explore **each branch** as **deep** as possible before **backtracking**.

Pseudocode

```
function DFS(G = ⟨V, E⟩, s ∈ V)
    for all v ∈ V do
        state[v] ← undiscovered,    parent[v] ← ∅
    end for
    state[s] ← root
    DFS_visit(G, s, parent, state)
    Return parent
end function

function DFS_VISIT(G = ⟨V, E⟩, u ∈ V, parent, state)
    state[u] ← discovered
    for all v ∈ Neighbors(u) do
        if state[v] = undiscovered then
            parent[v] ← u
            DFS_visit(G, v, parent, state)
        end if
    end for
end function
```

Example



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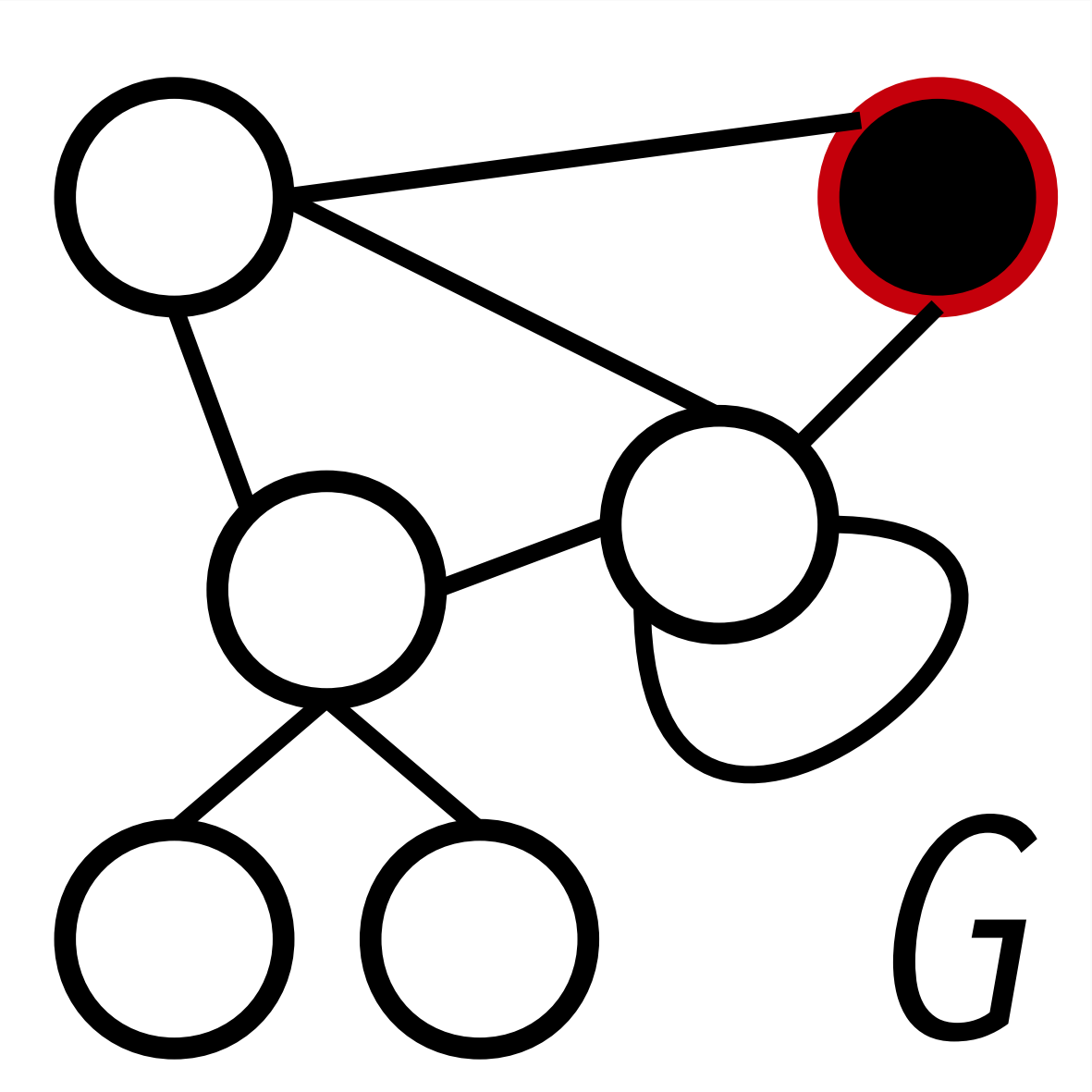

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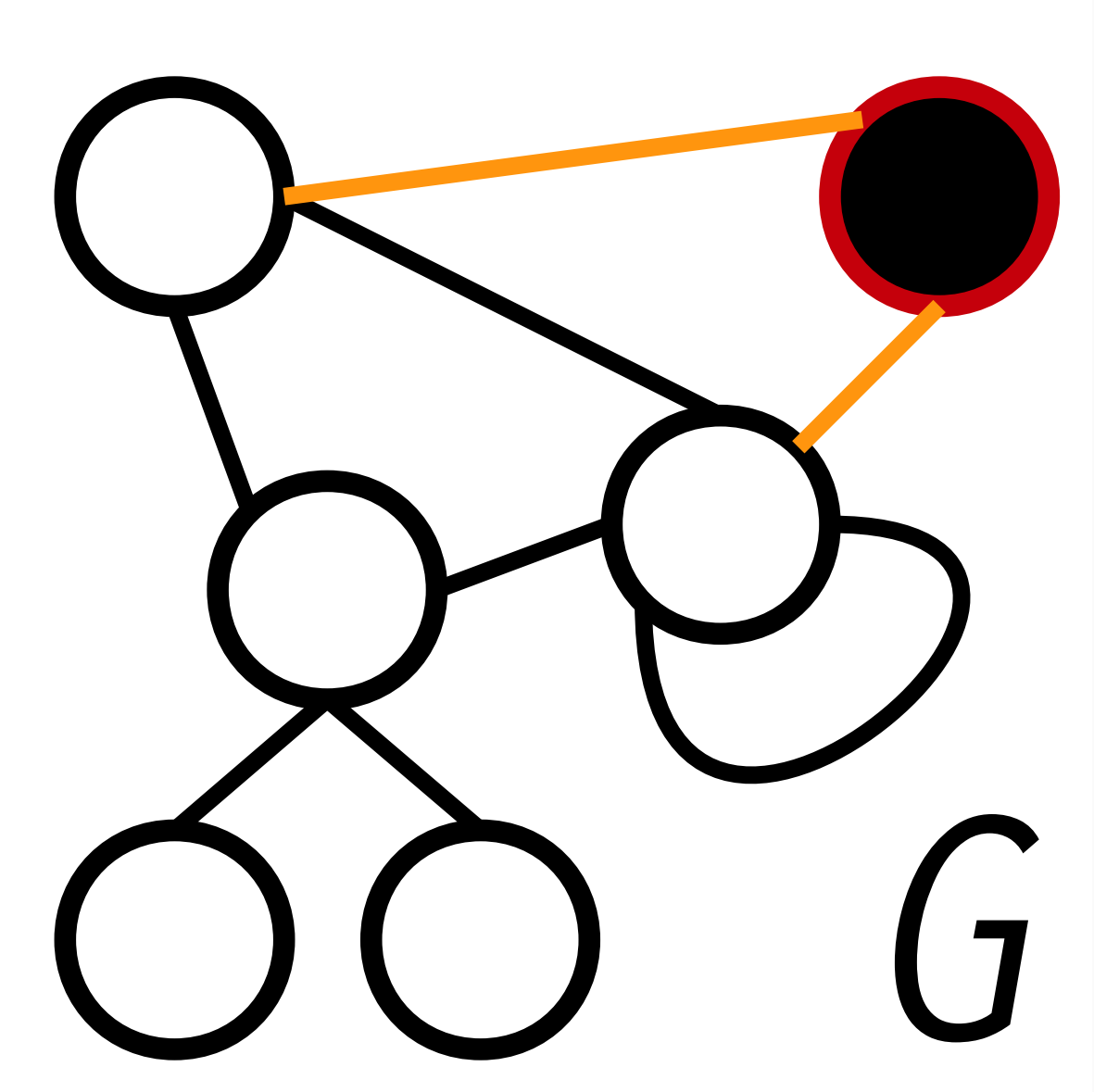

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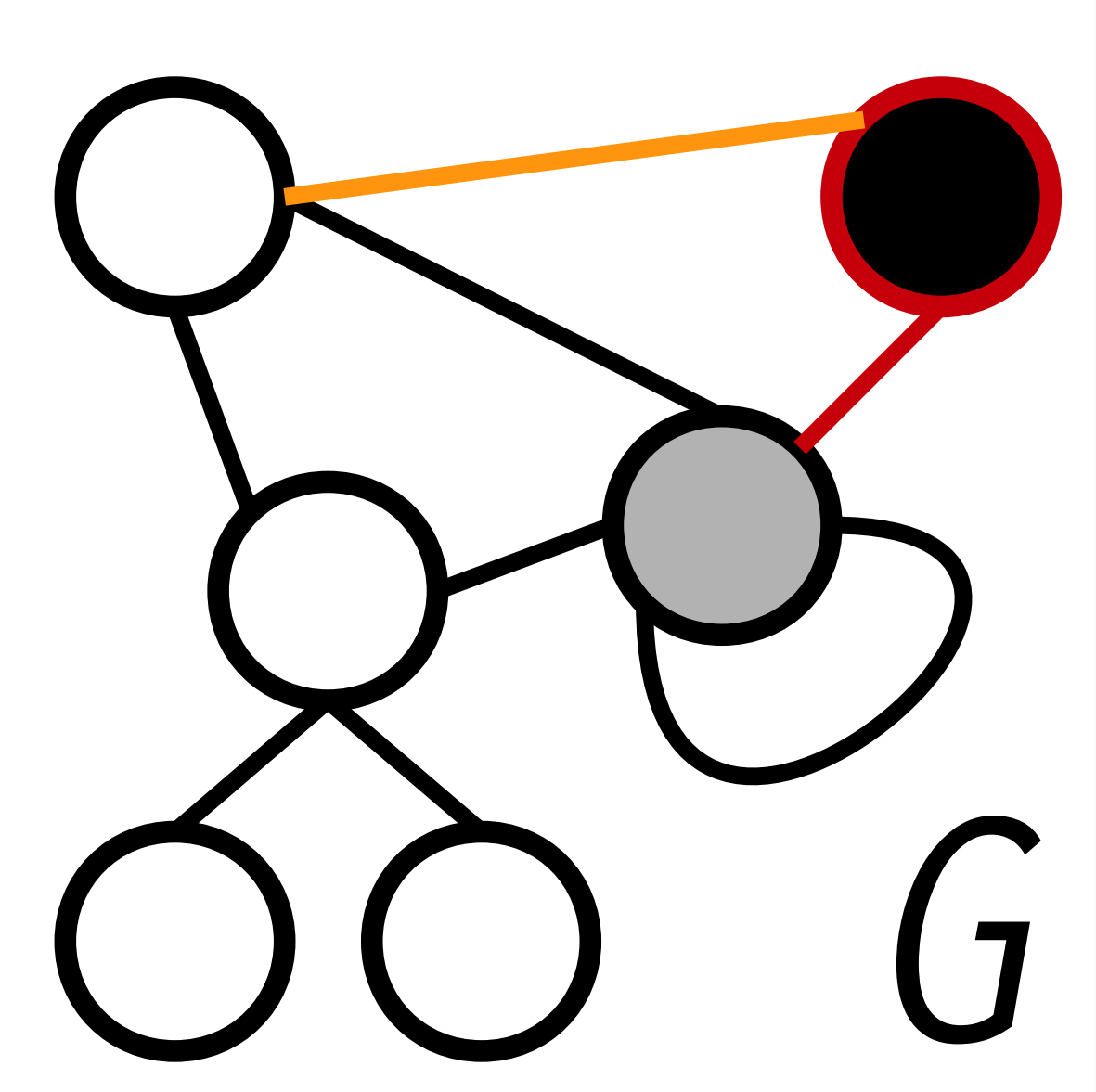

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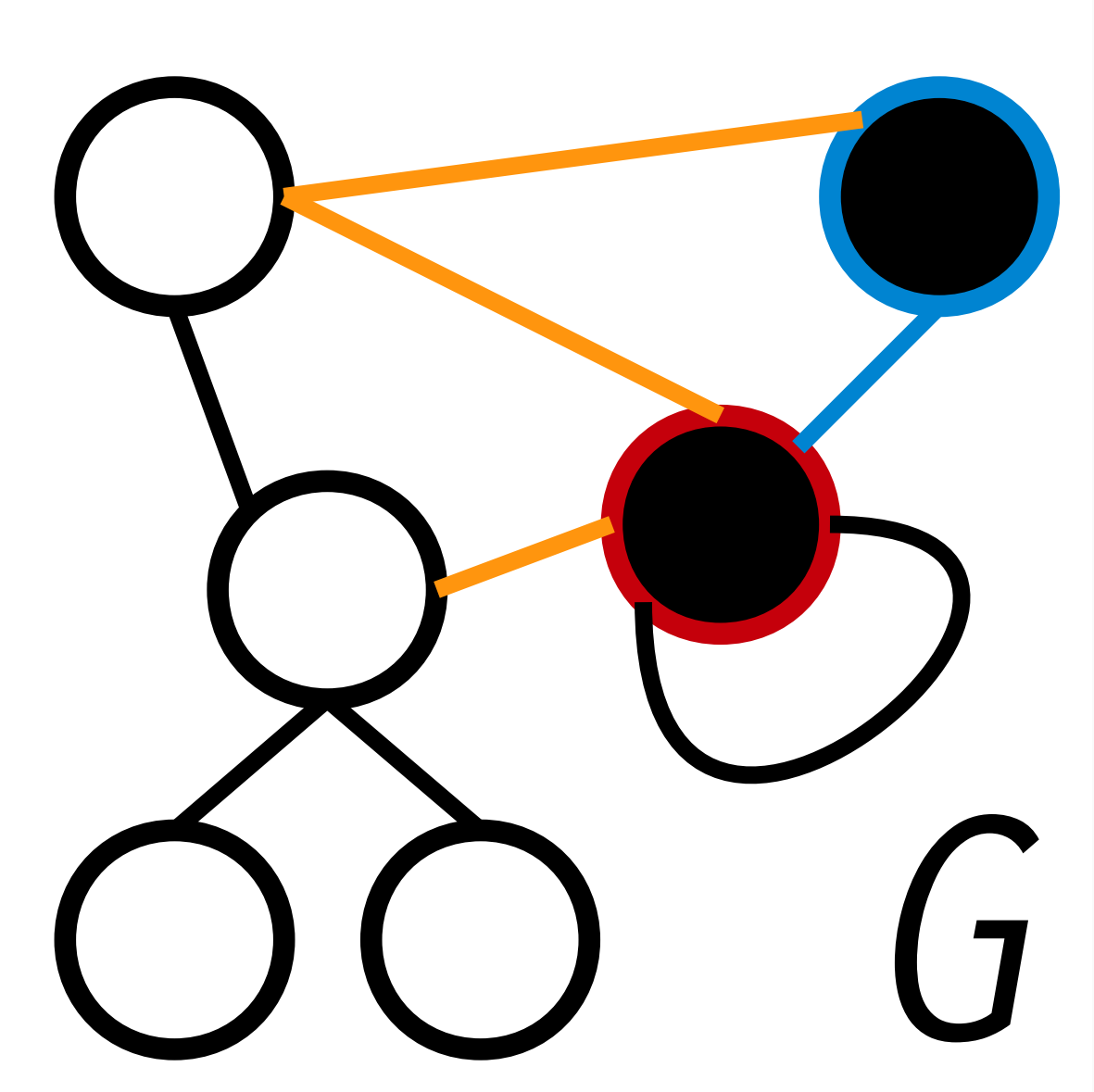

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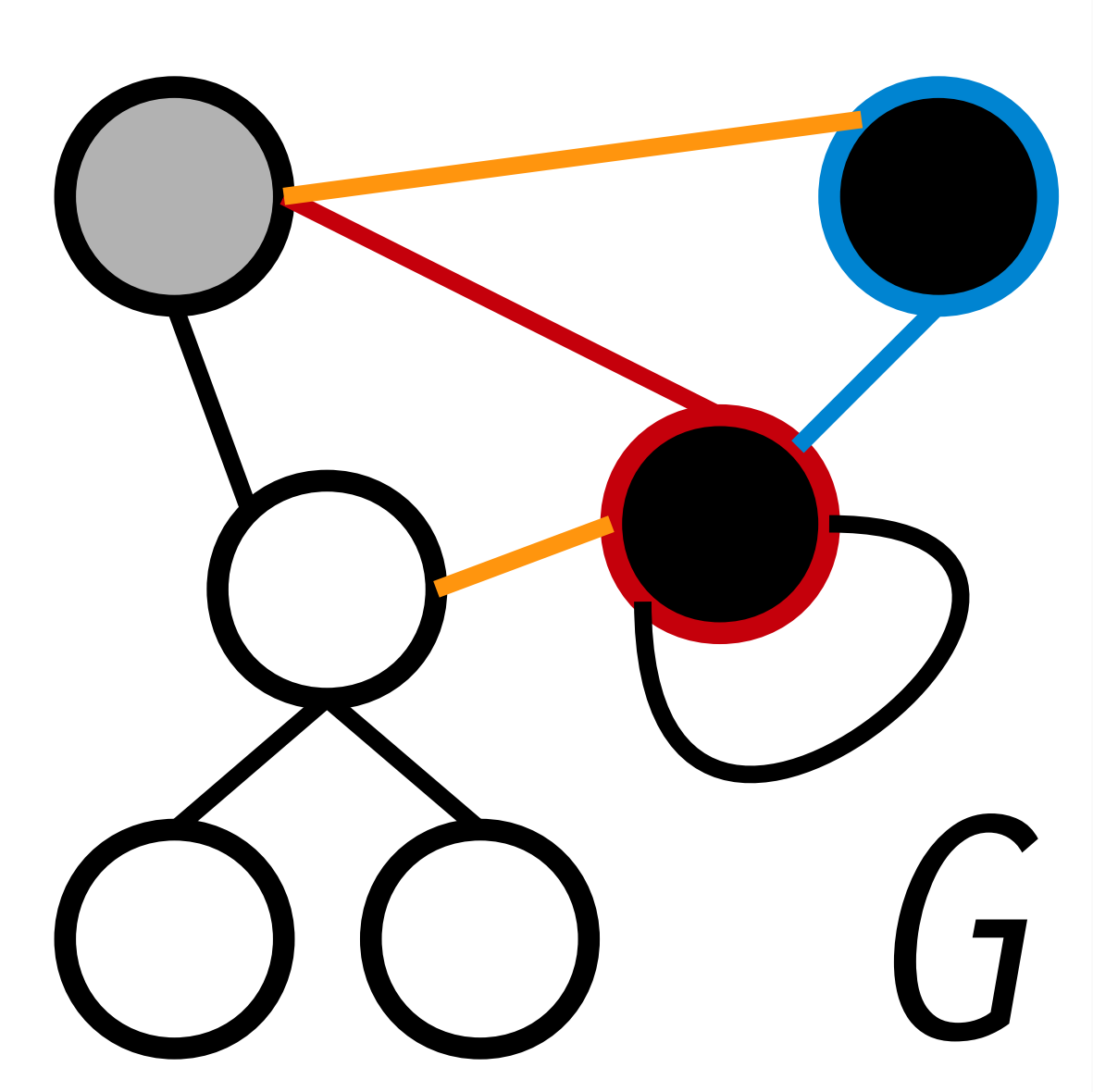

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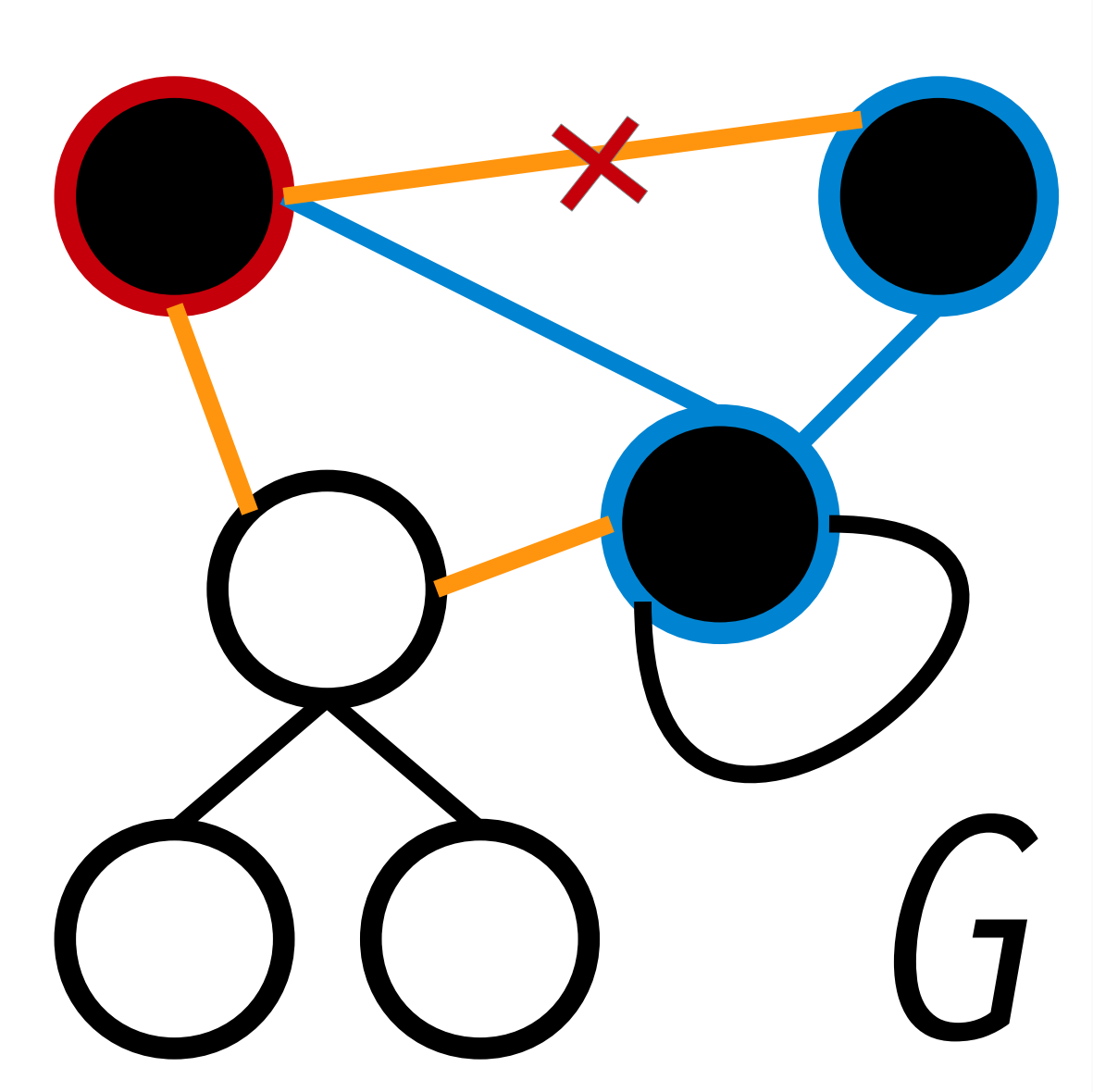

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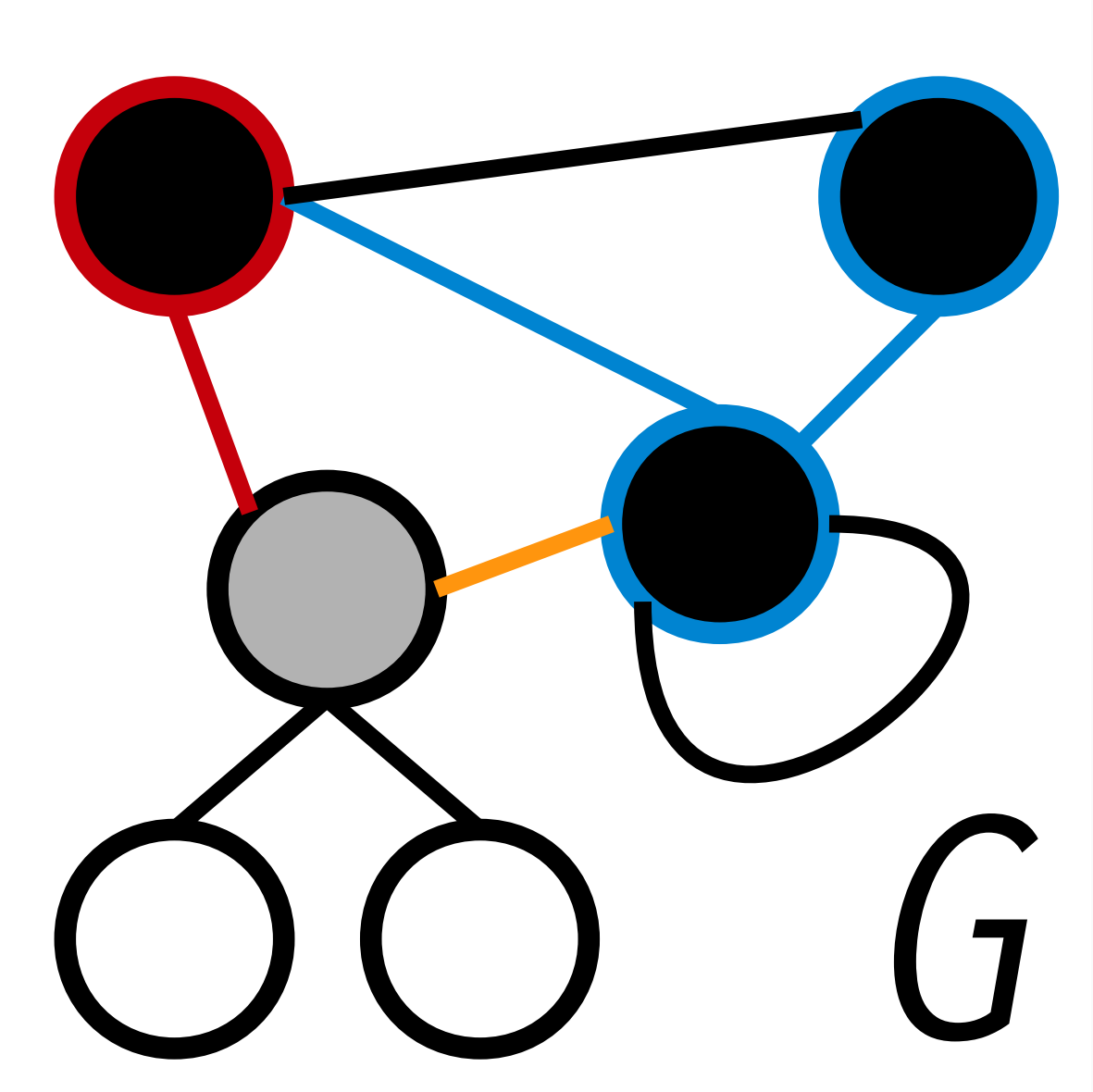

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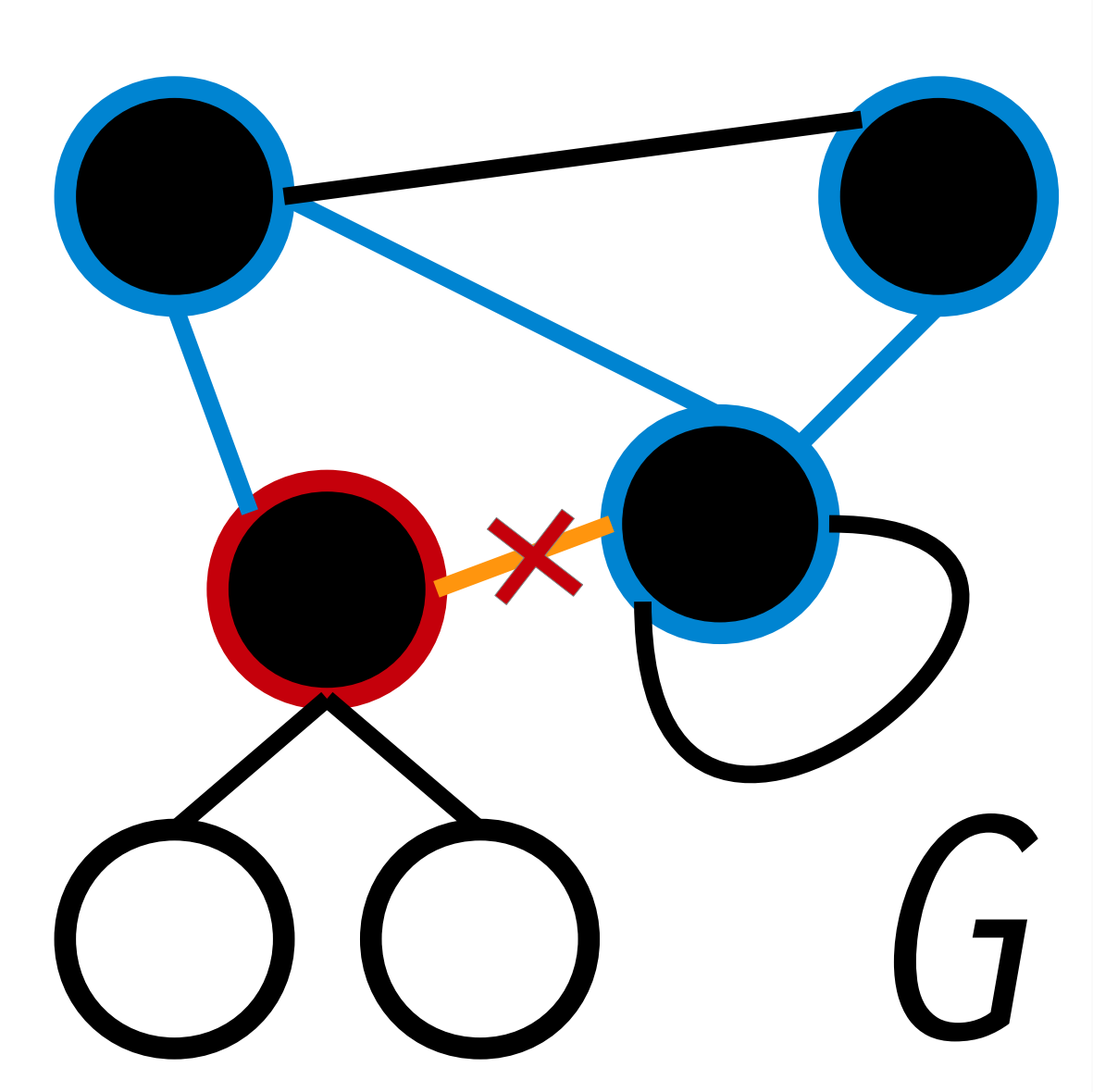

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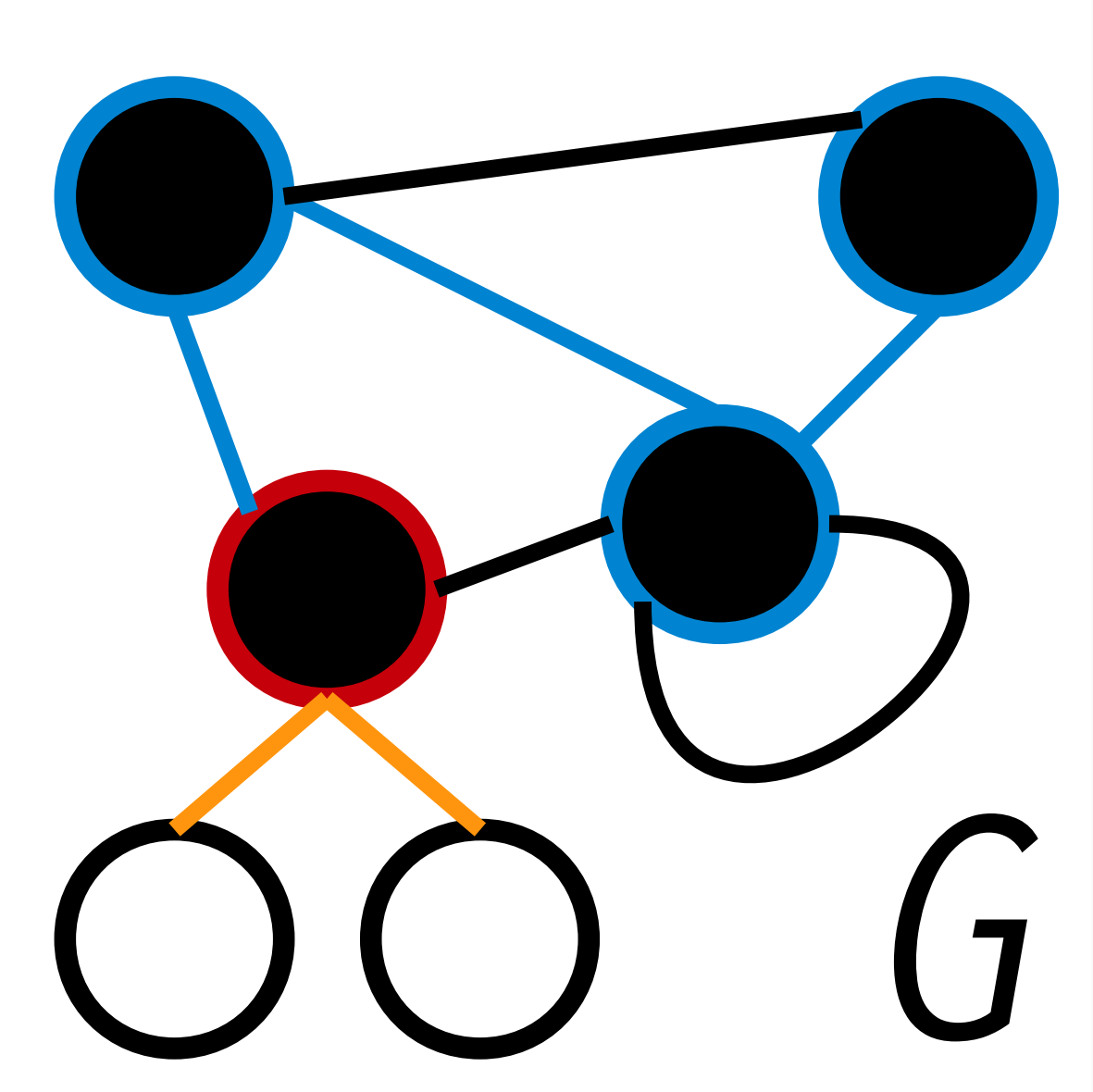

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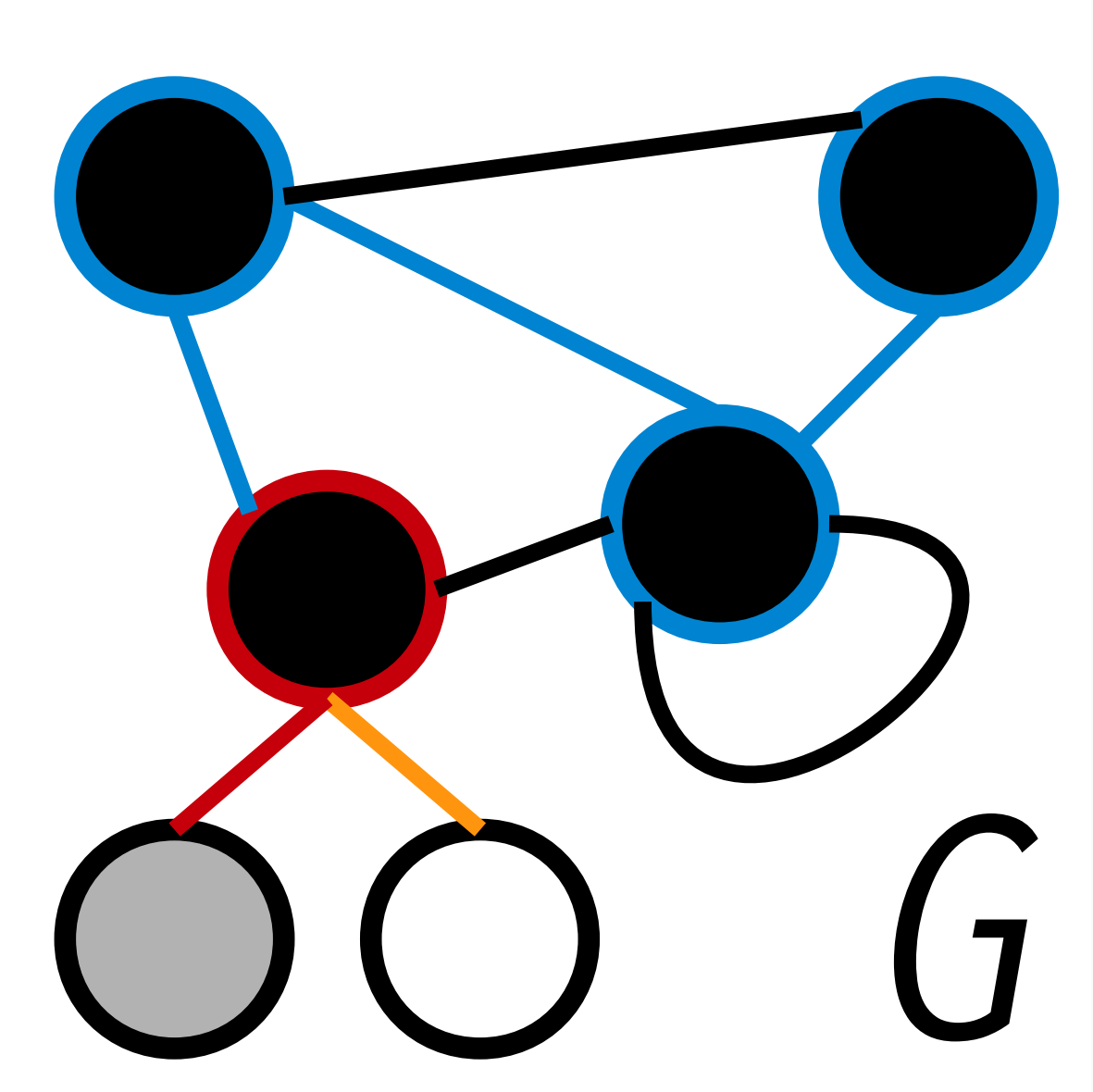

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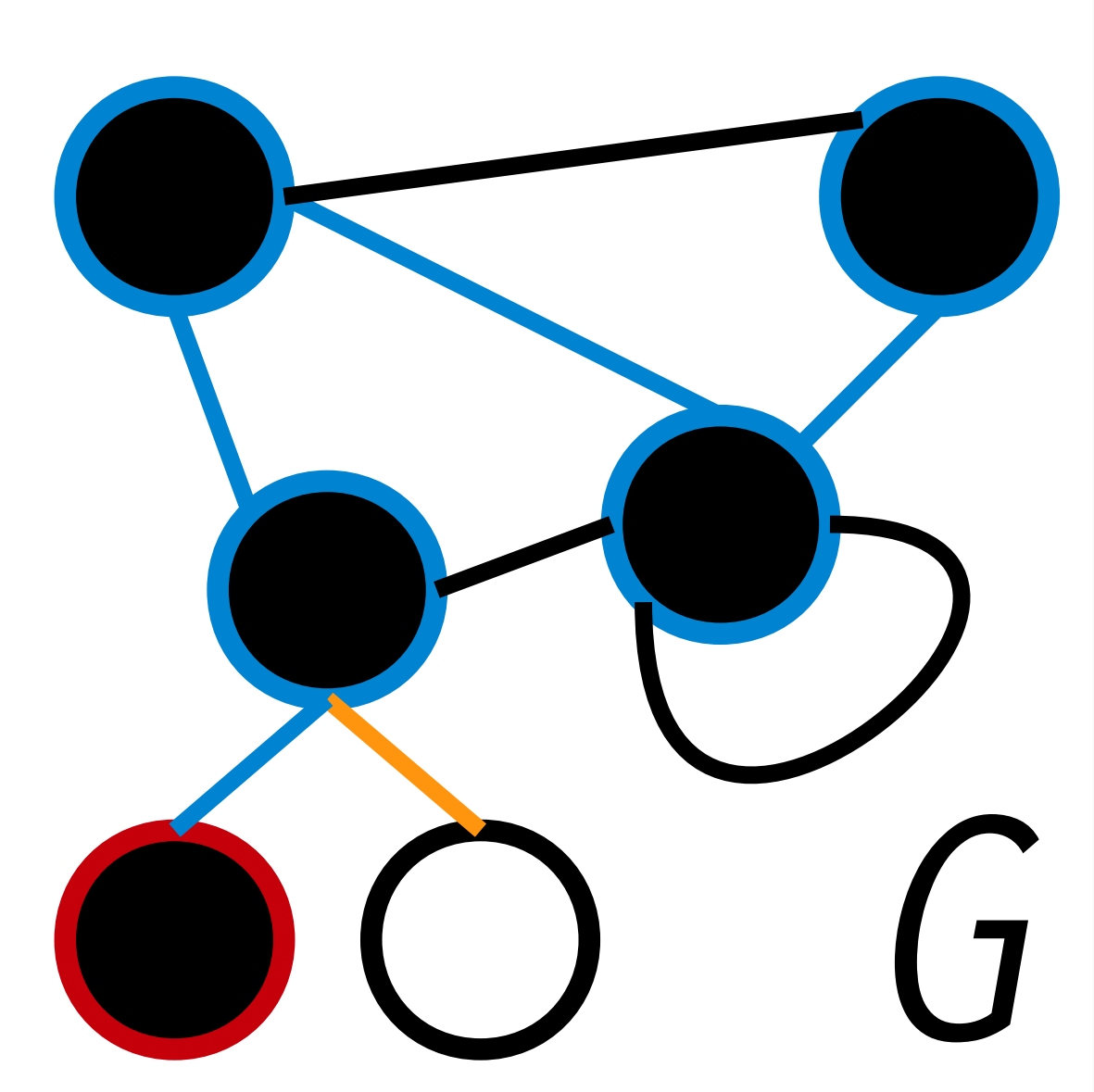

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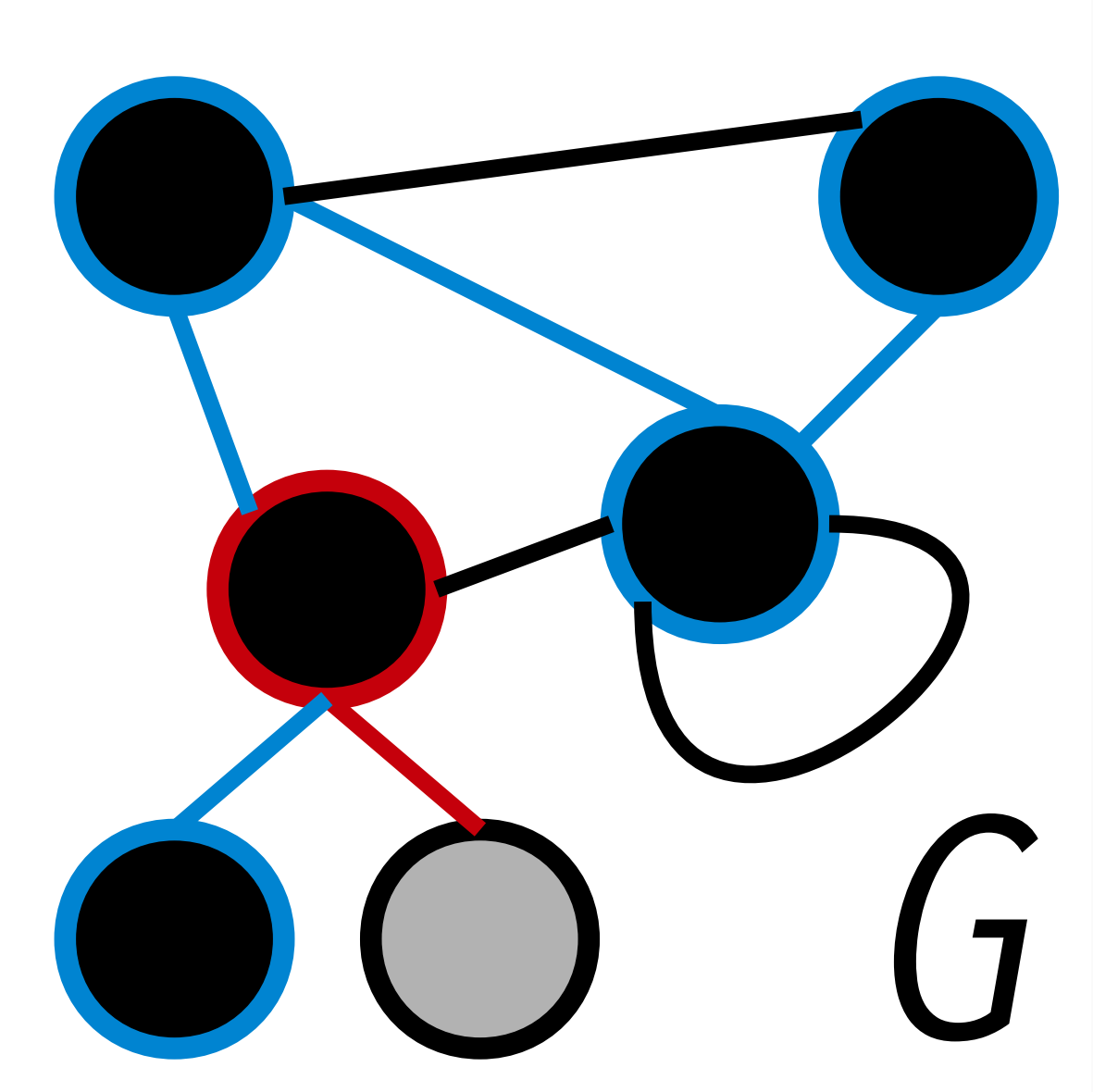

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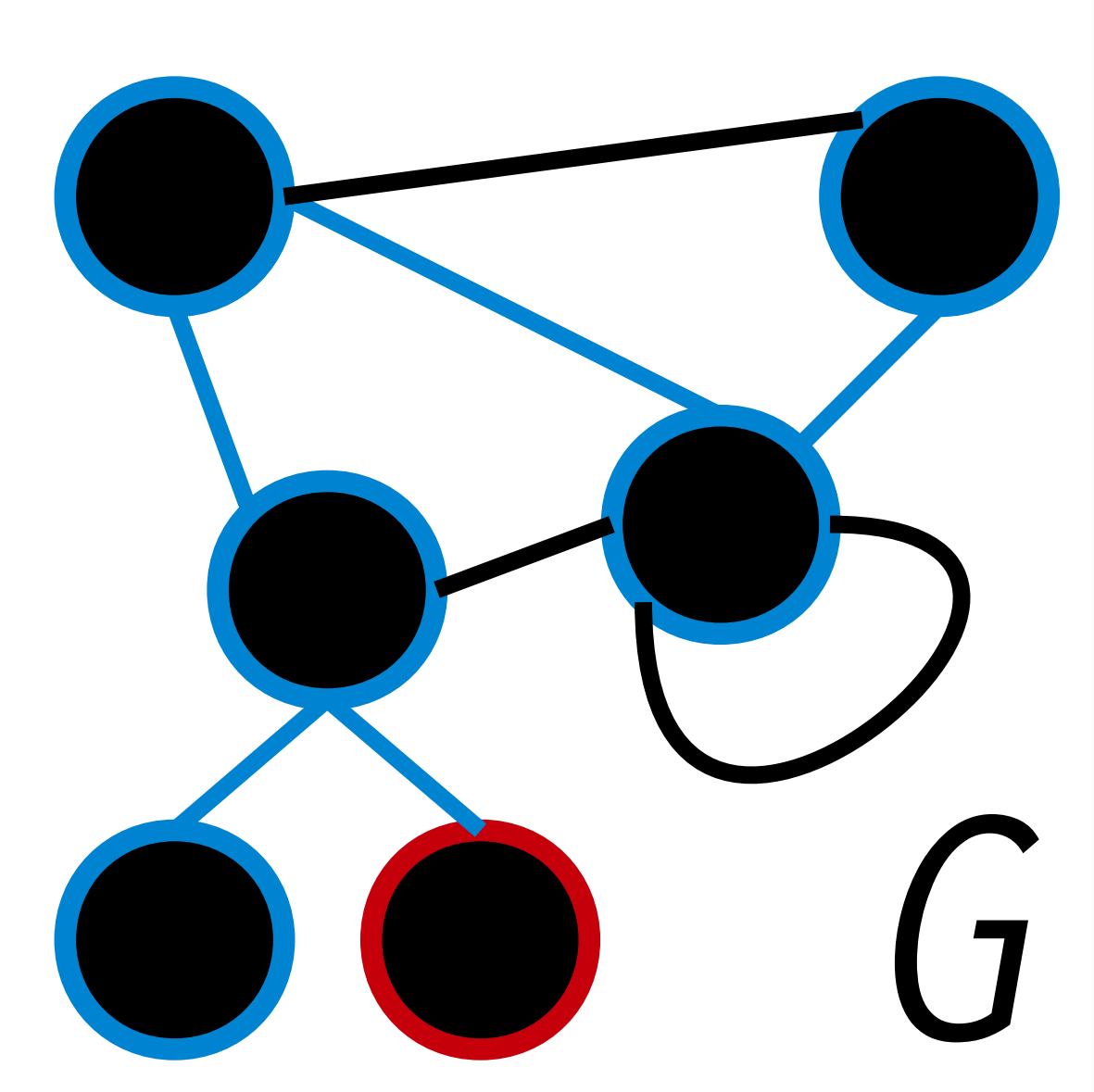

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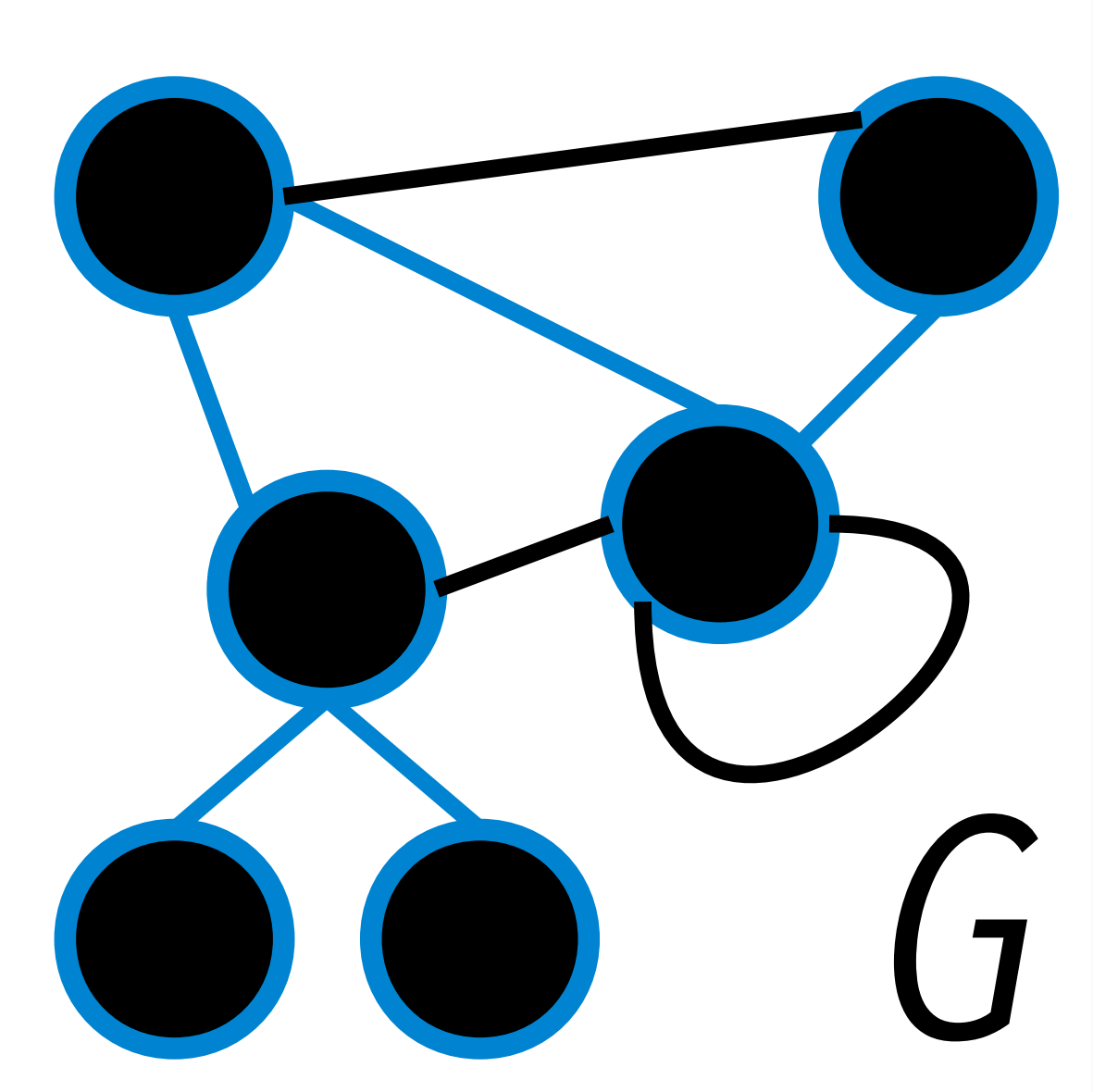


Analysis

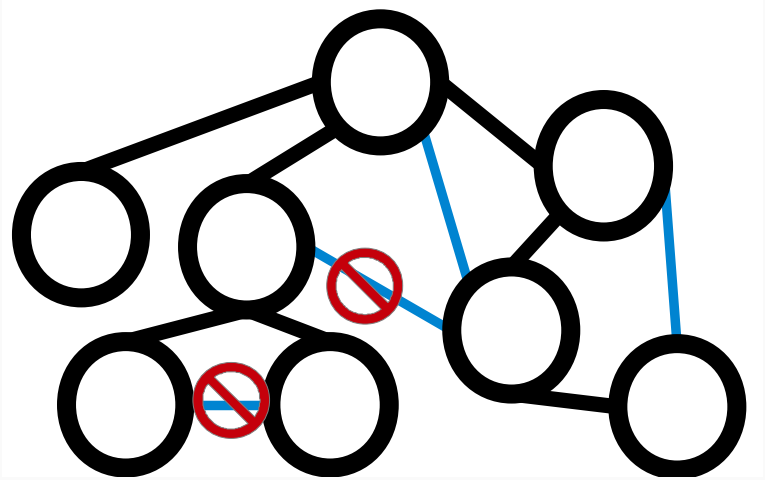
Property 1: DFS $\rightarrow$ spanning tree $T = \langle V, E' \rangle$ (pseudocode).

Time Complexity: $\mathcal{O}(|E| + |V|)$ (Why?)

Property 1: DFS $\rightarrow$ Nodes **ordering**

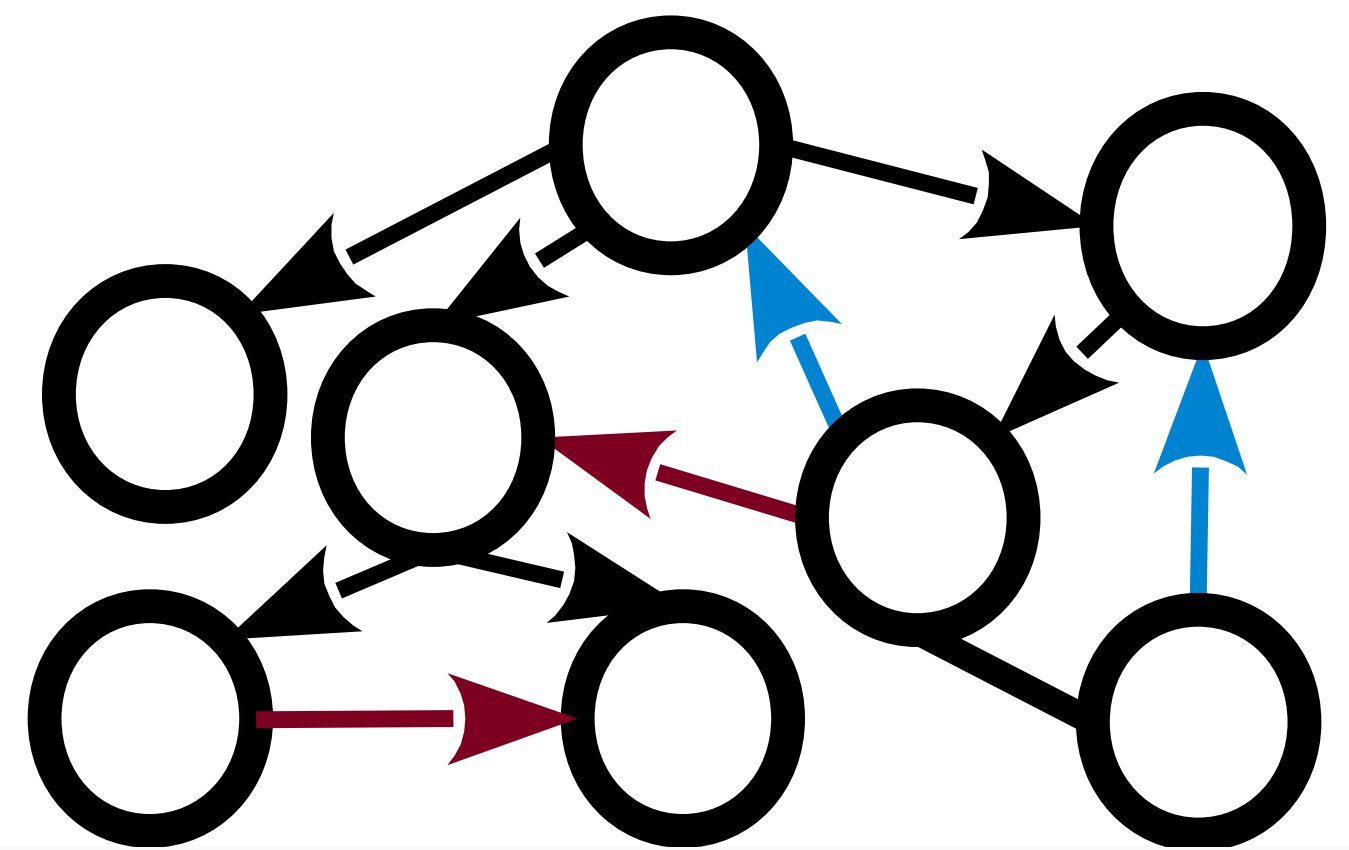
Property 2: DFS $\rightarrow$ **biconnectivity**, **bridges**, **cut nodes**.

Property 3: (Undirected graphs) edge e s.t. $e \in E$ and $e \notin E'$; e links a node with an ancestor in T; e$\neq$cross-edge (Proof?).



Backward-edges, Forward-edges, Cross-edges

- Forward-edges : black
- Backward-edges : blue
- Cross-edges : red

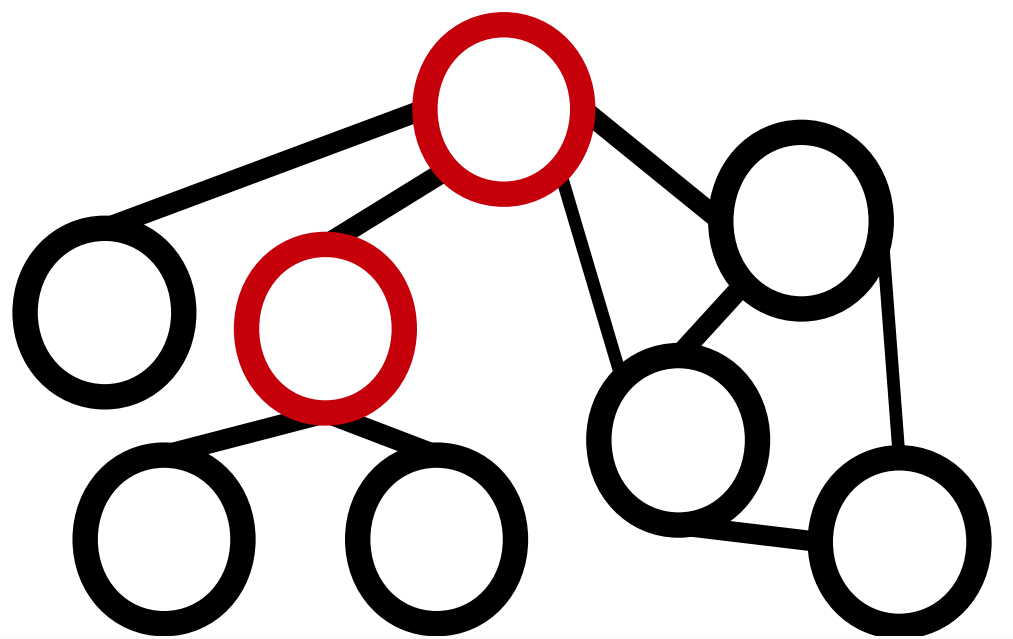


Biconnectivity using DFS

Biconnectivity

Definition 1: Graph $G = \langle V, E \rangle$ is **biconnected** if it has **no cut/articulation node**, i.e., 2 nodes or more should be removed to disconnect G.

Definition 2: A **cut/articulation node** is a node $v \in V$ s.t. removing v disconnects G.



Biconnectivity | Algorithm

Easy solution: Remove each node and test with BFS or DFS is G remains connected (Complexity?).

Analysis:

- **Root:** Cut node if its nb. of children ≥ 2
- **Leaves:** Never
- **Internal node** u: cut node if $\exists$ subtree rooted at a child of u **without** back-edges to an ancestor of u.

Articulation node detection

Methodology:

- Keep track of **back-edges going higher**; using **DFS Discovery times** (*discT*).
- $l(v)$: **Highest** node **reached** from **subtree rooted** at v

$l(v) = min\{discT[w] : u \in Descendent(v) \text{ and } \langle u, w \rangle \text{: back-edge}\}$

u is an **articulation point** if $\exists v \in Children(u)$ s.t.:

$$l(v) \geq discT[u]$$

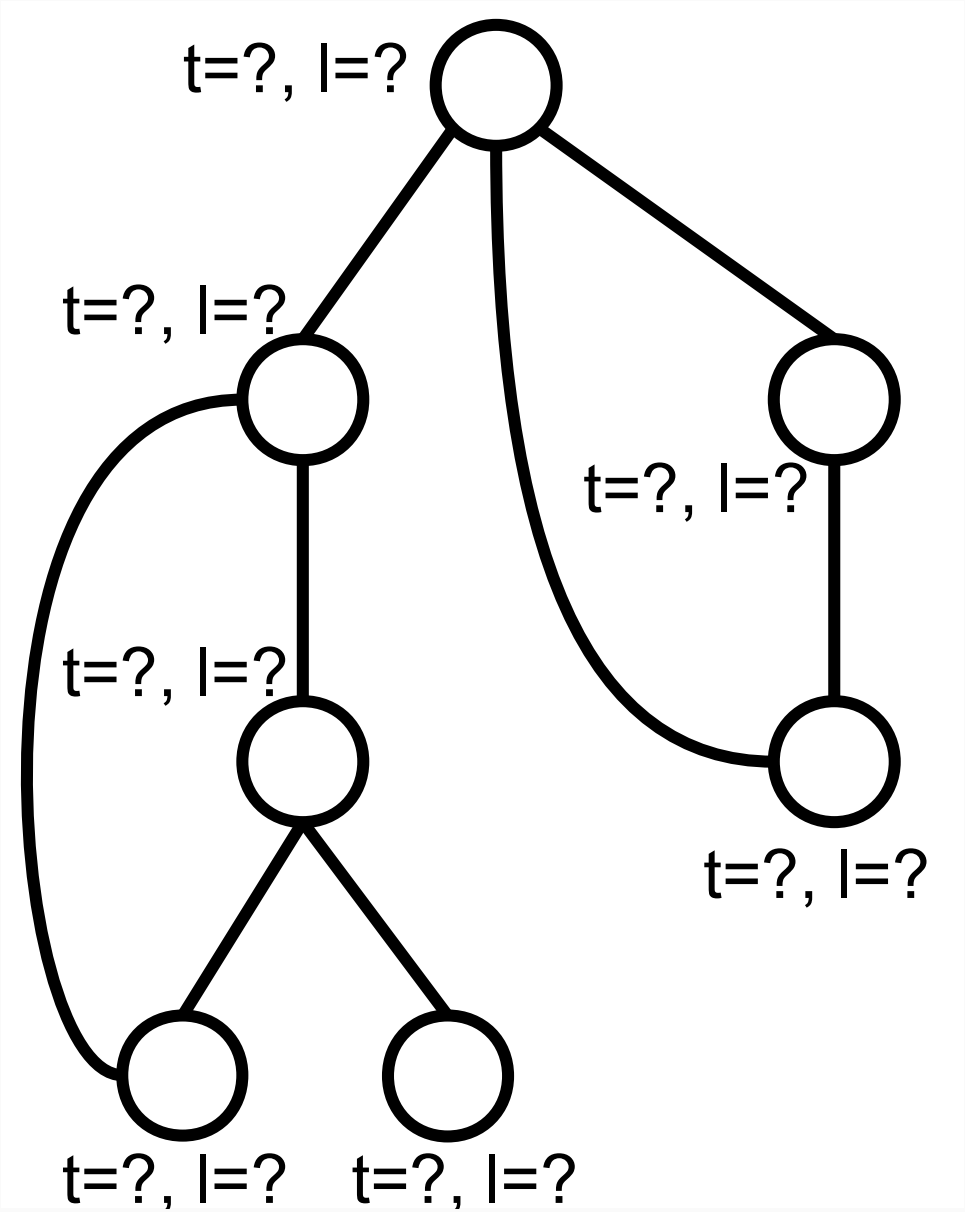
Pseudocode | Compute $l(u), \quad \forall u \in V$

```
function DFS_L(G, u, parent, state, l, t, discT)
    state[u] ← discovered
    discT[u] ← time;    l[u] ← time
    t ← t + 1
    for all v ∈ Neighbors(u, G) do
        if state[v] = undiscovered then
            parent[v] ← u
            DFS_l(G, v, parent, state, t)
            // After DFS_l l(v) is stored in l[v]
            l[u] ← min(l[u], l[v])
        else
            if v ≠ parent[u] then
                // Then ⟨u, v⟩ is a back-edge
                l[u] ← min(l[u], discT[v])
            end if
        end if
    end for
end function
```

Pseudocode | Retrieve Articulation Points

```
function ARTICULATION_POINTS(G, parents, T, l, discT)
    for all v ∈ V do
        if parent[v] = root then // v is the root
            if |Neighbors(v, T)| > 1 then
                articulation[v] ← true
            end if
        else
            for all u ∈ V  s.t.  parents[u] = v do
                if l[u] ≥ discT[v] then
                    articulation[v] ← true
                end if
            end for
        end if
    end for
end function
```

Example



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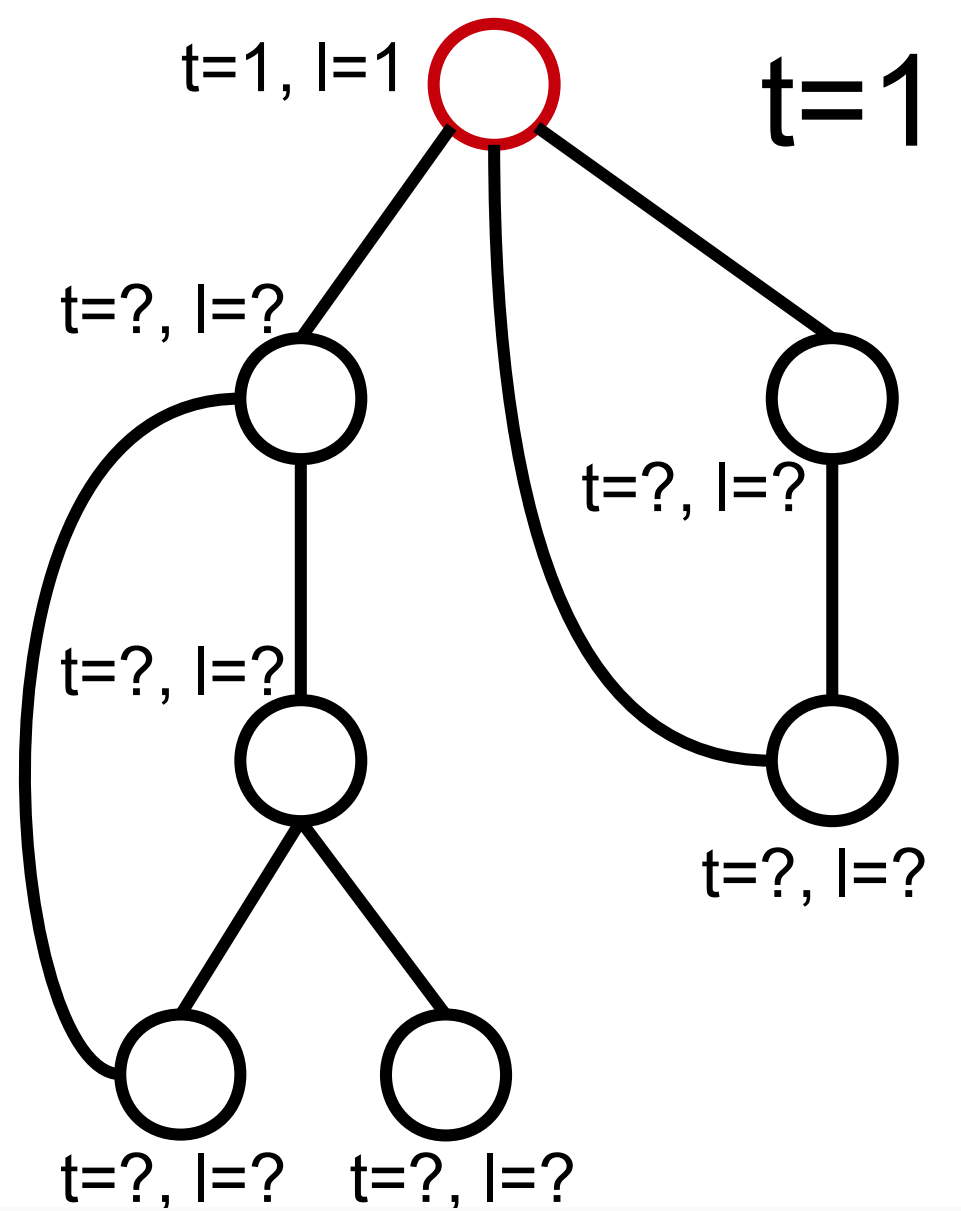

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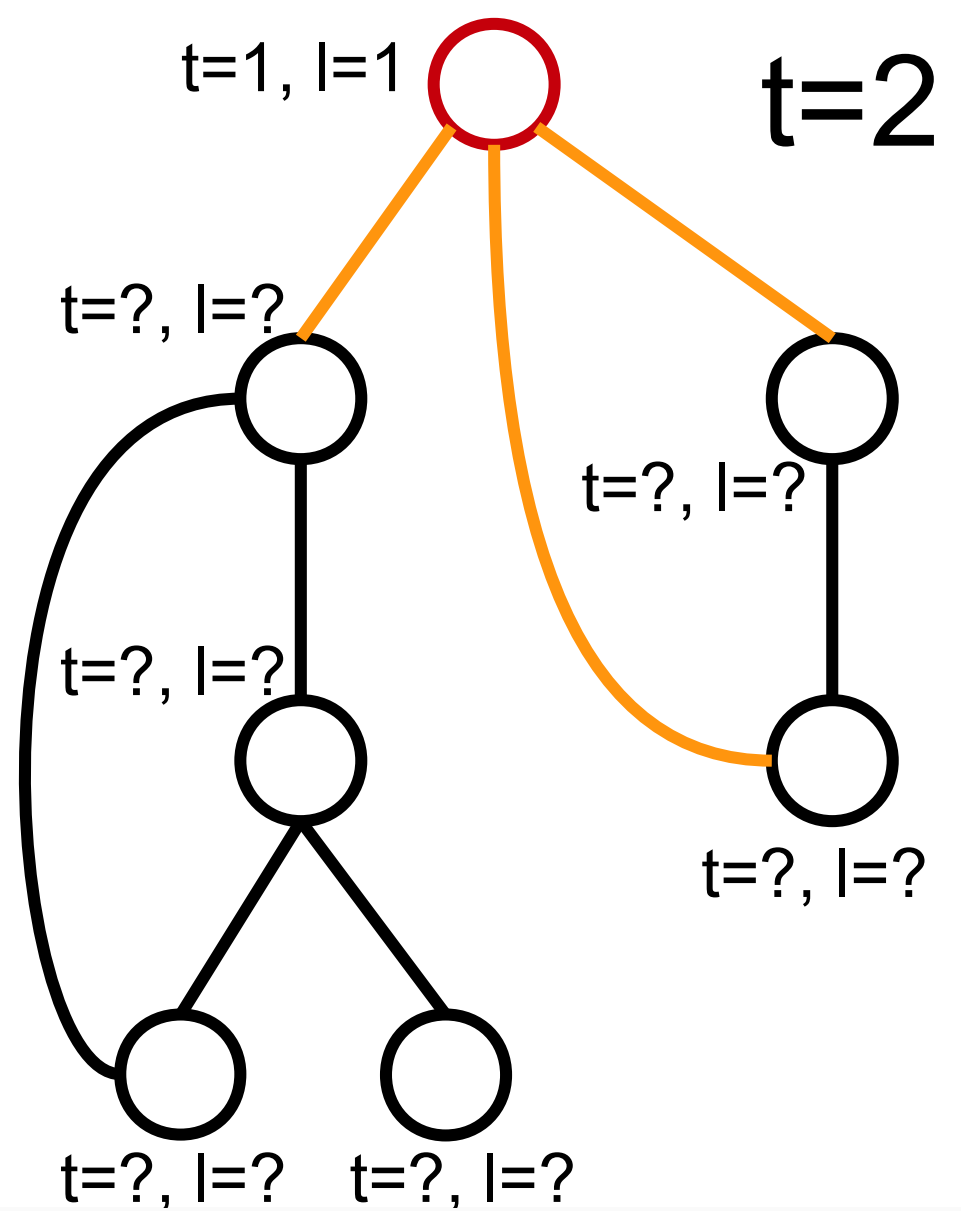

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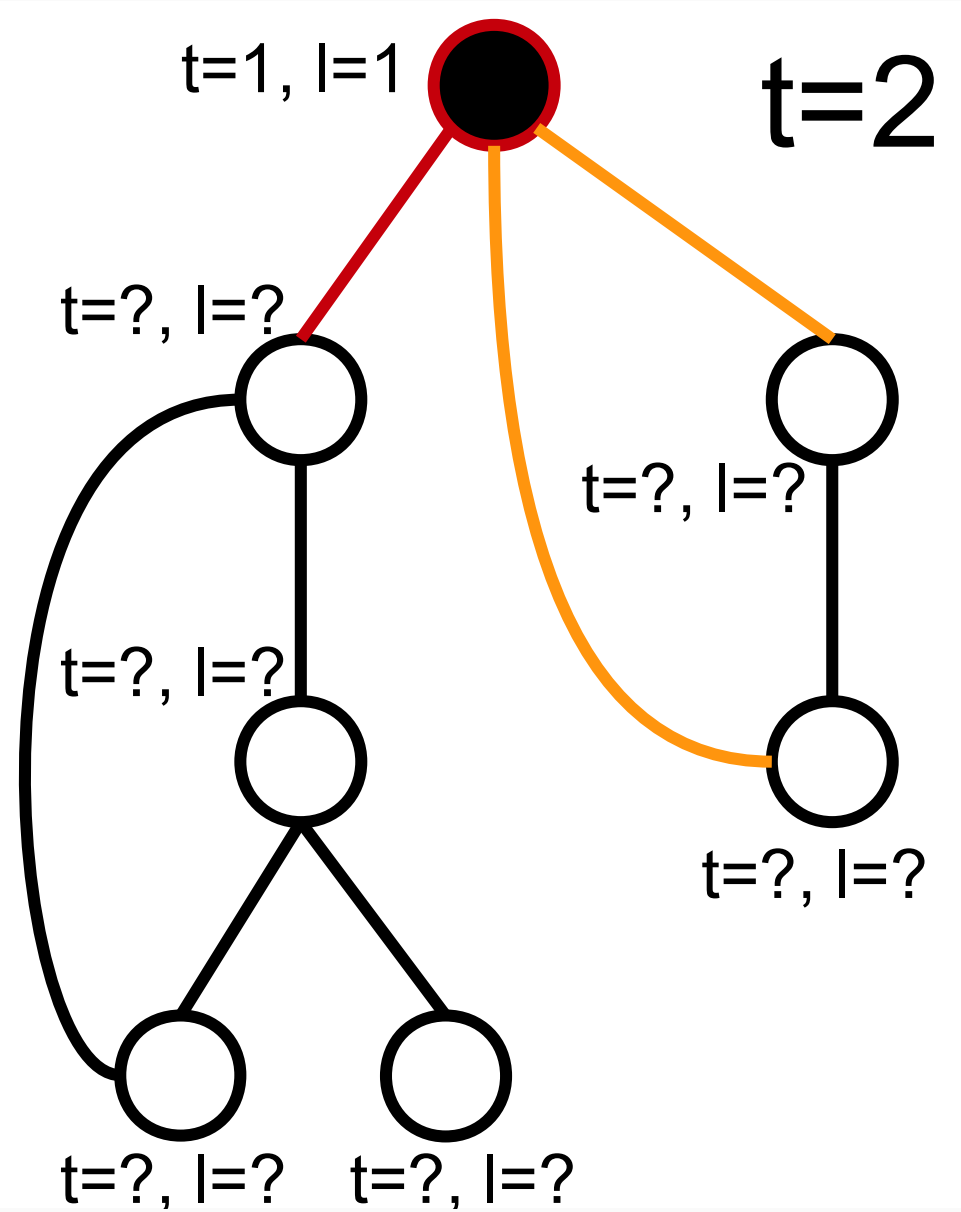

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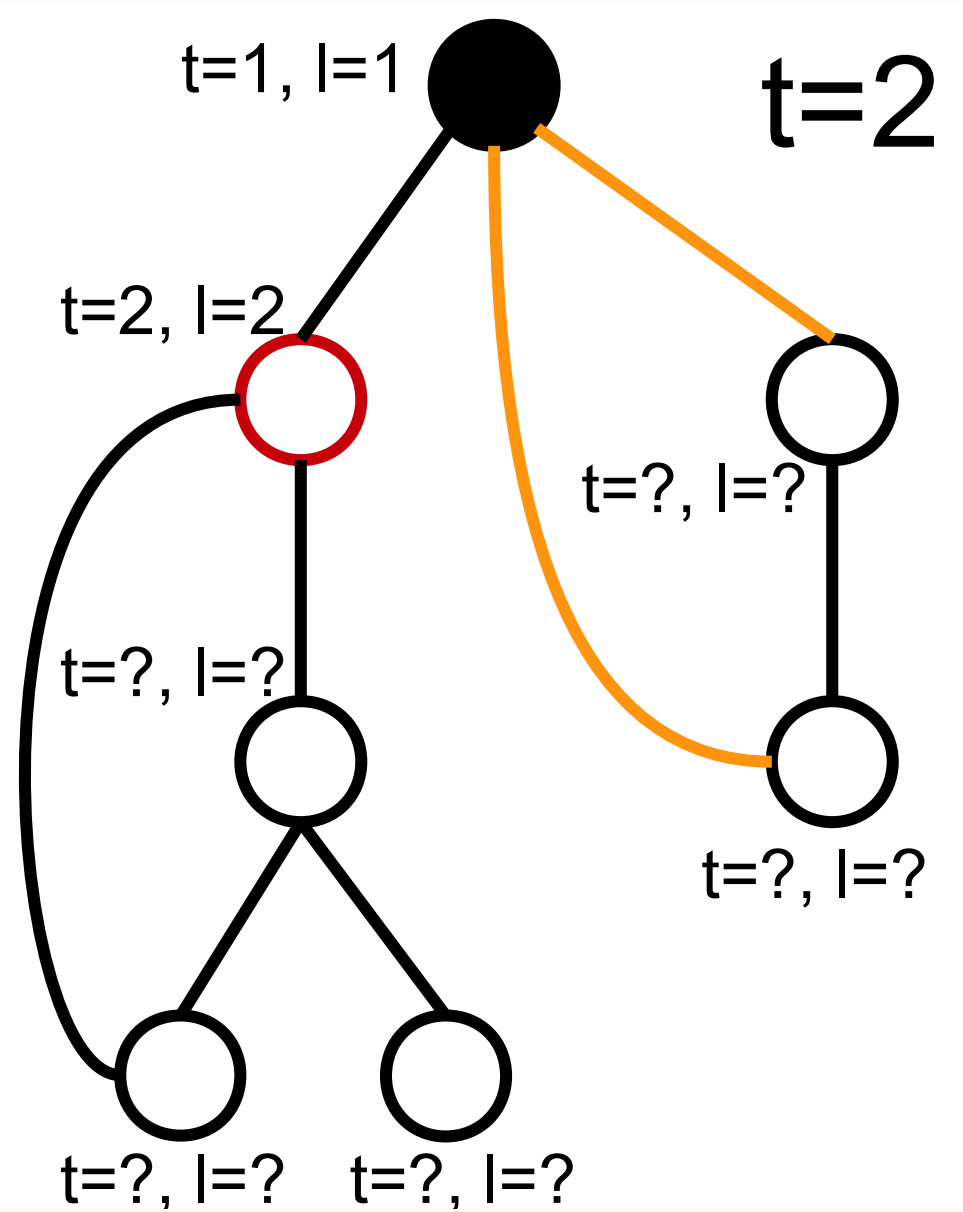

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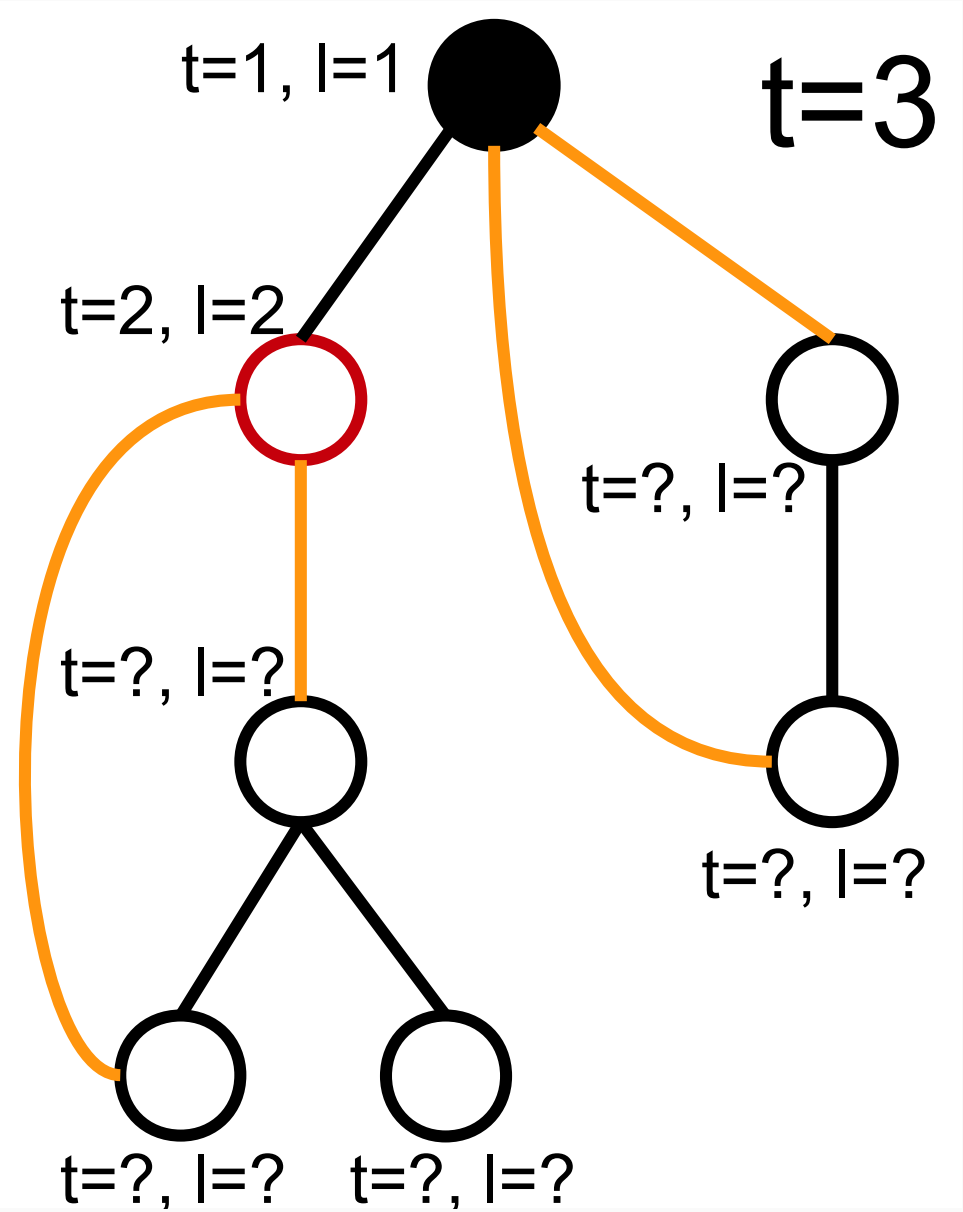

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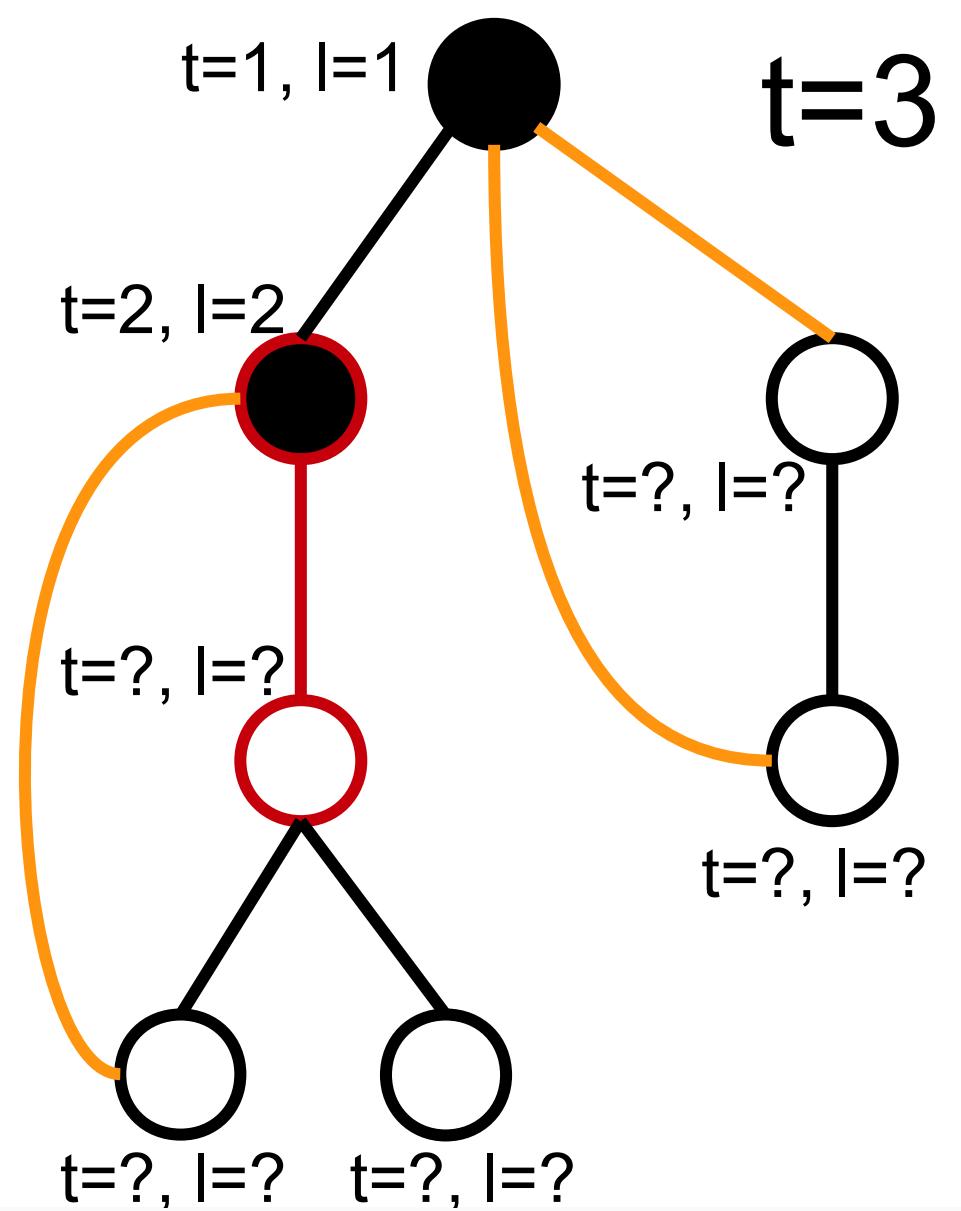

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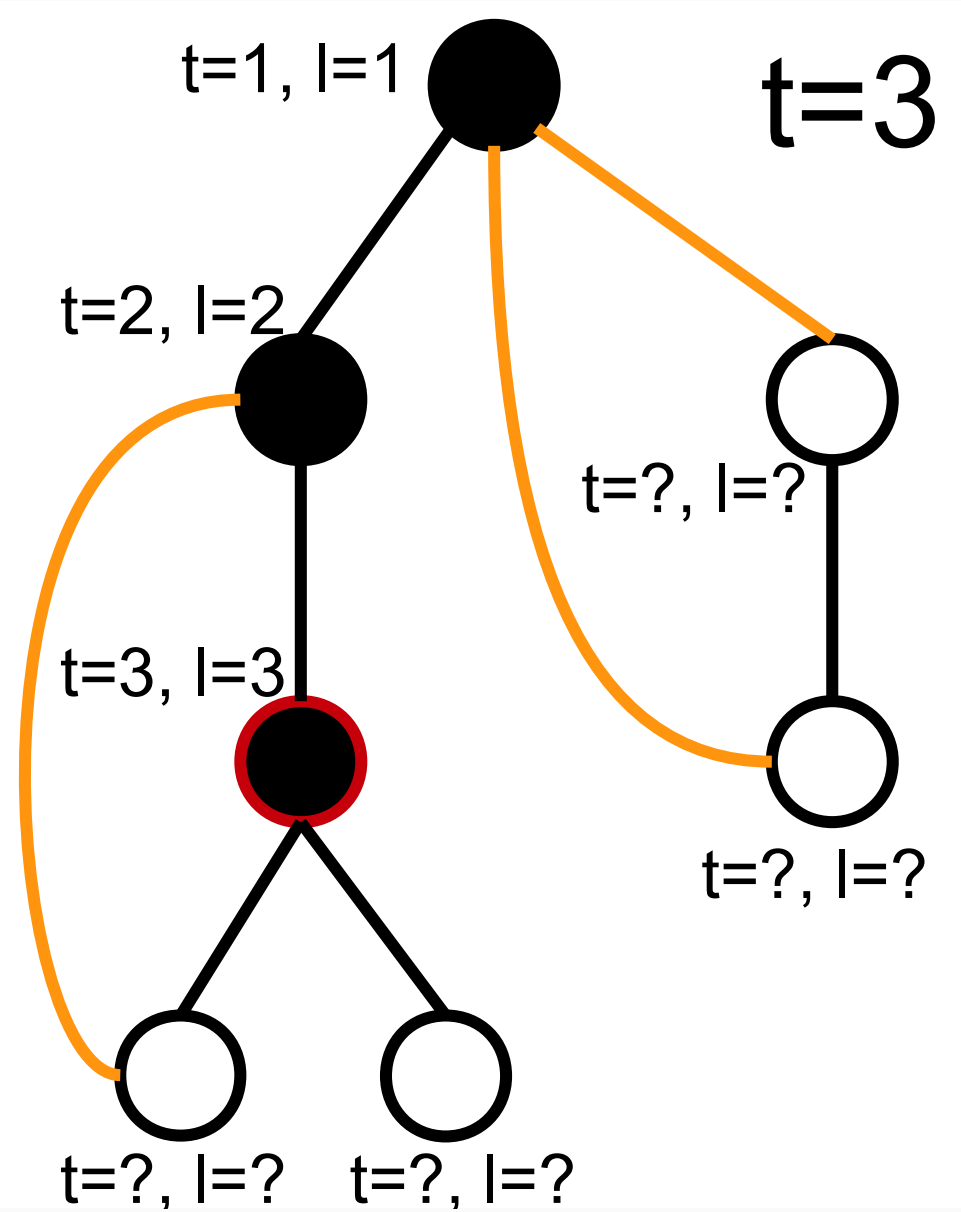

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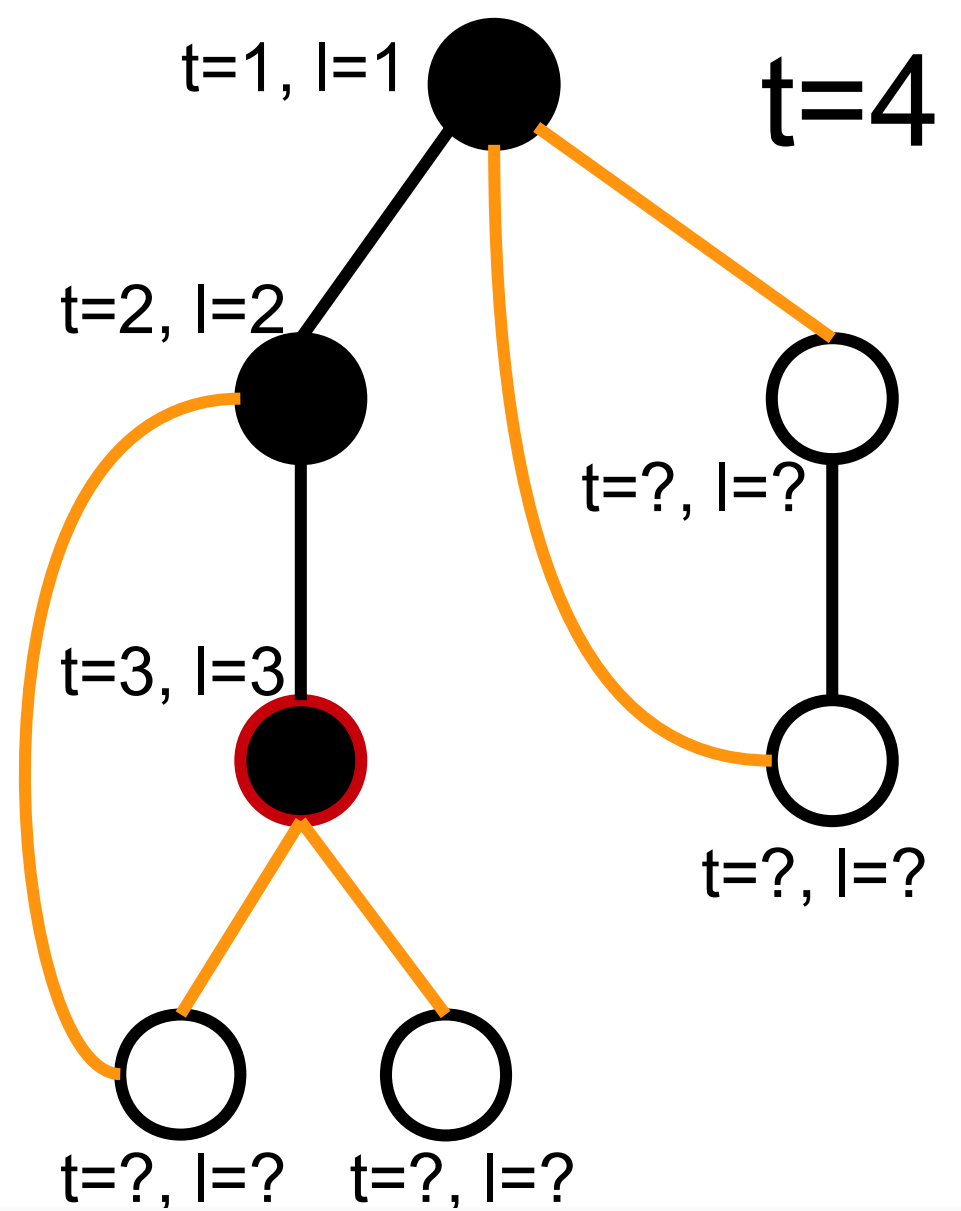

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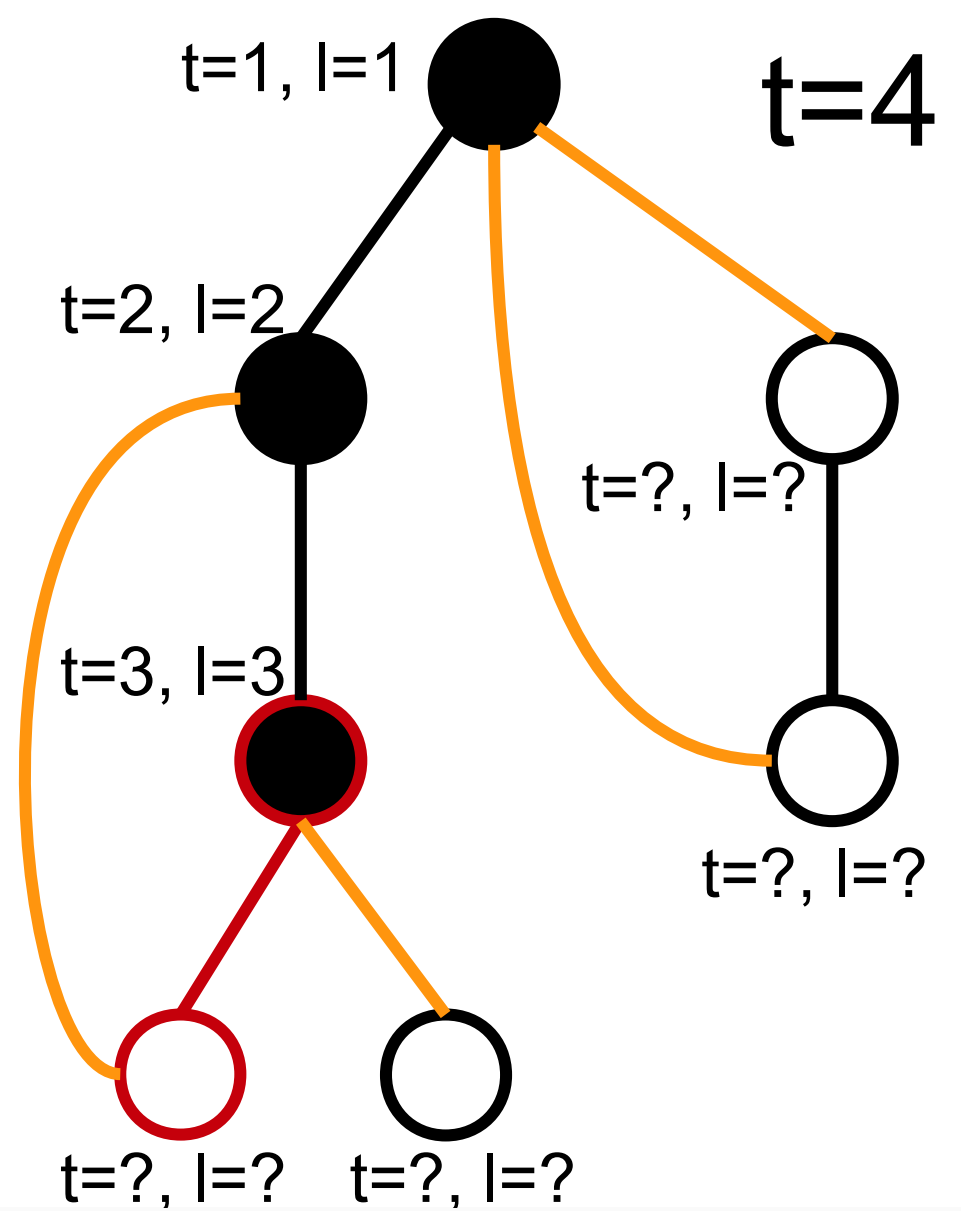

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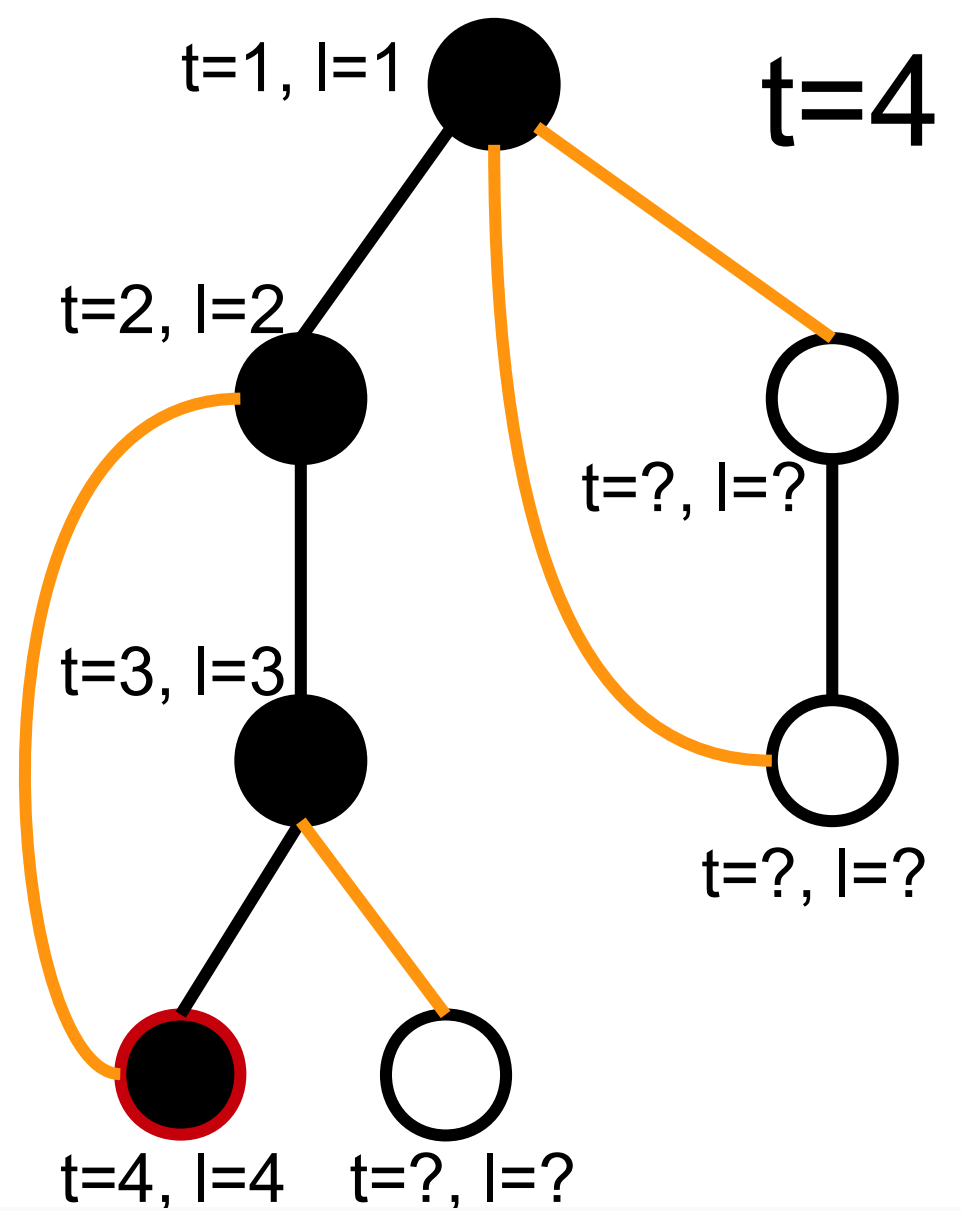

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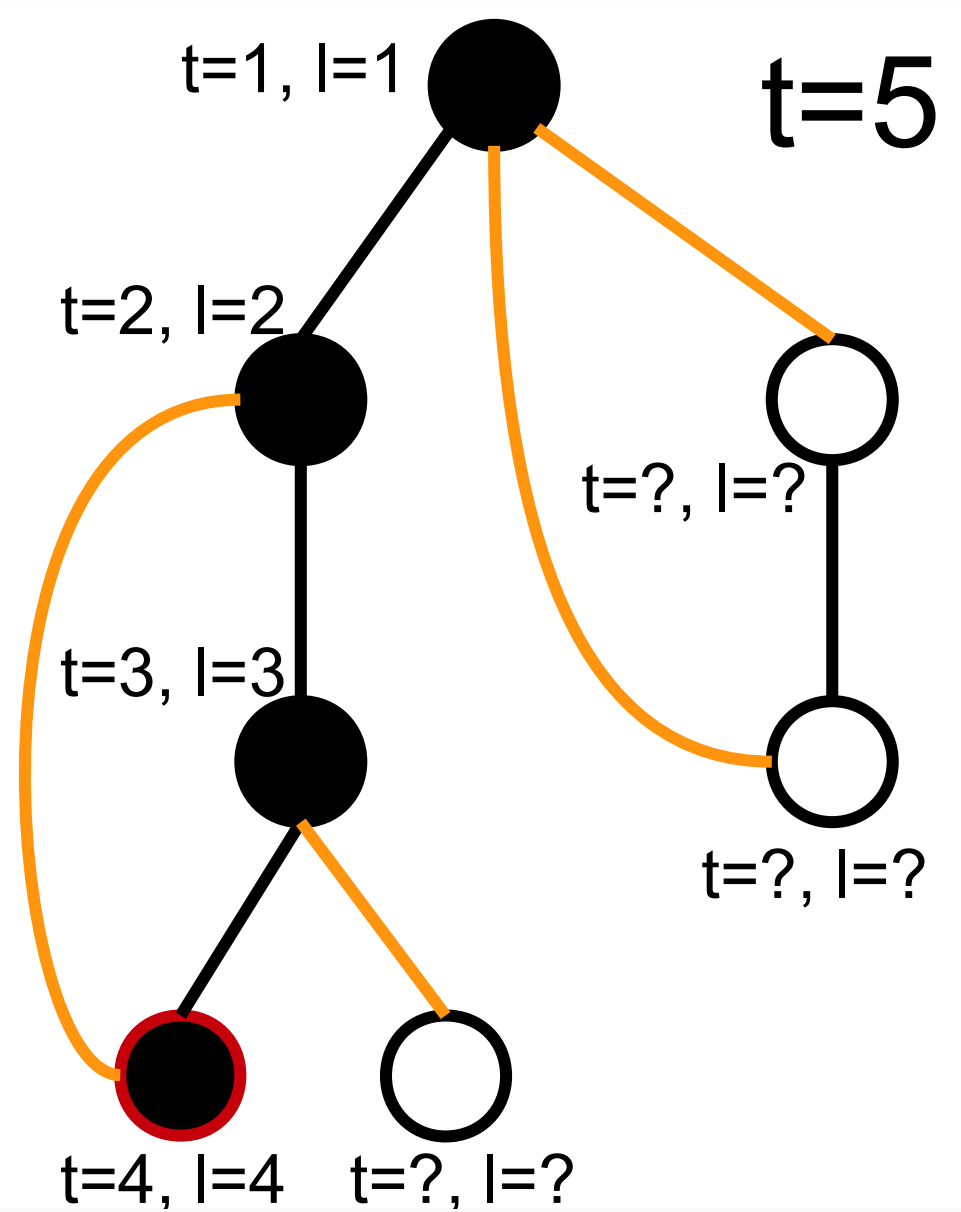

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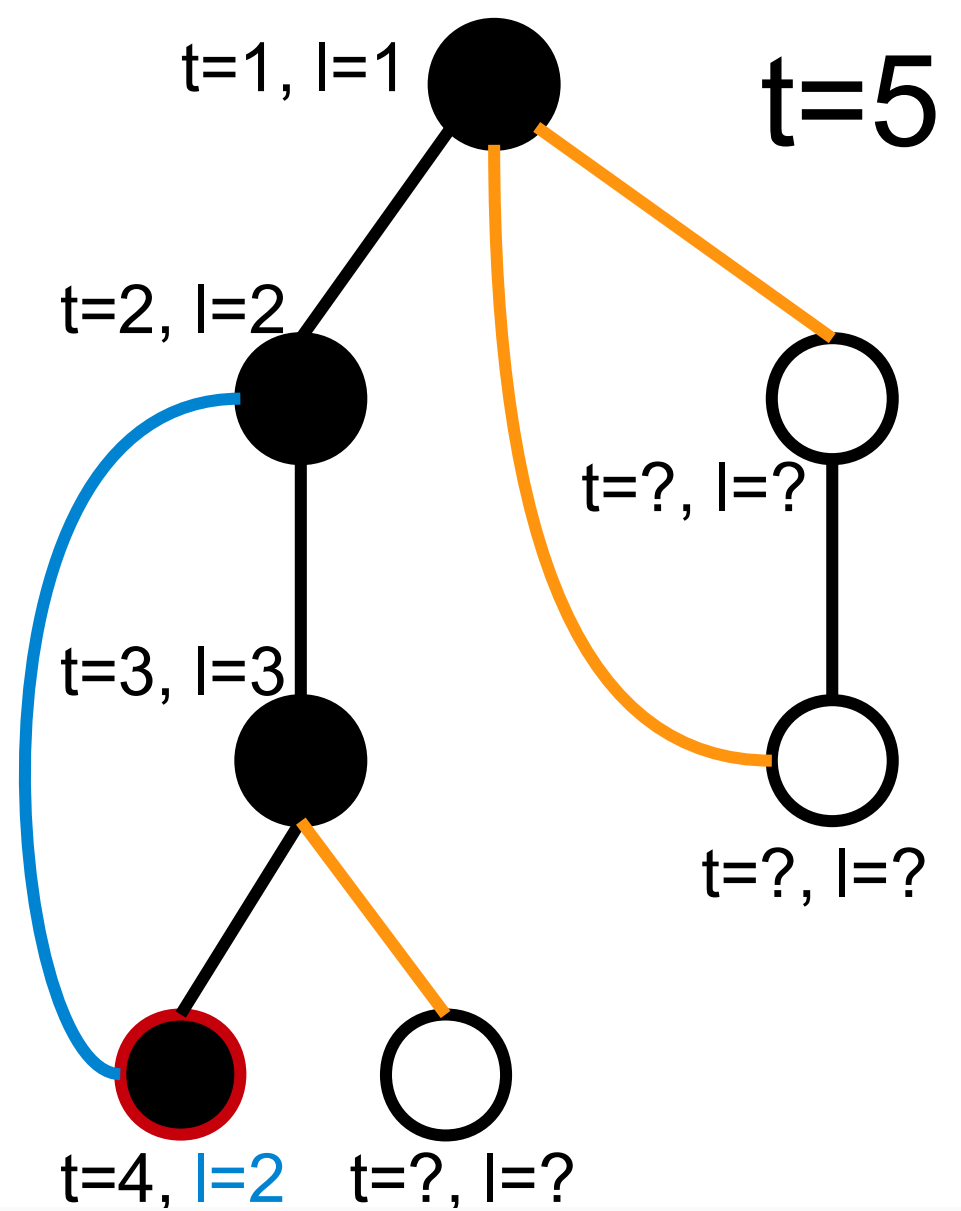
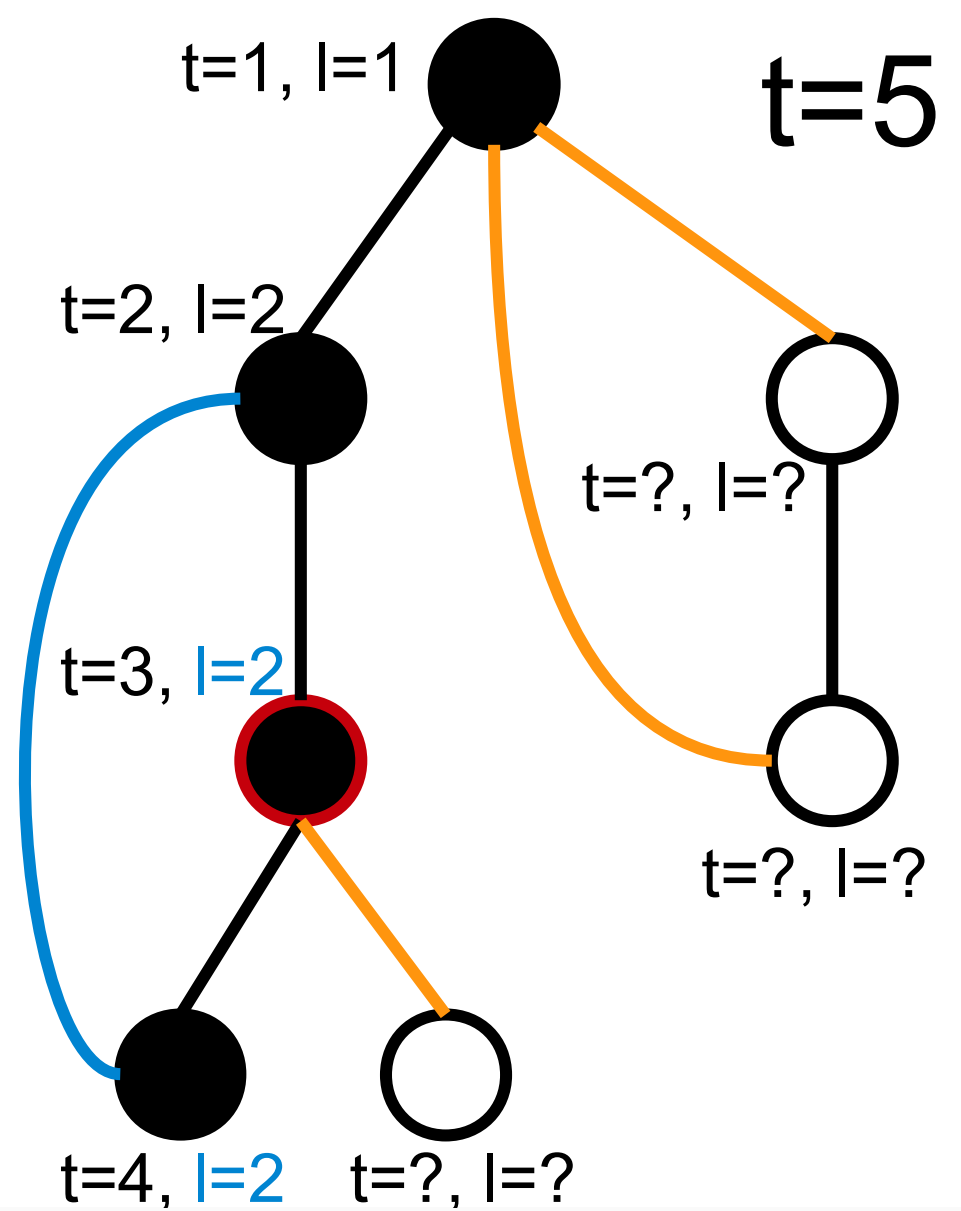


Example



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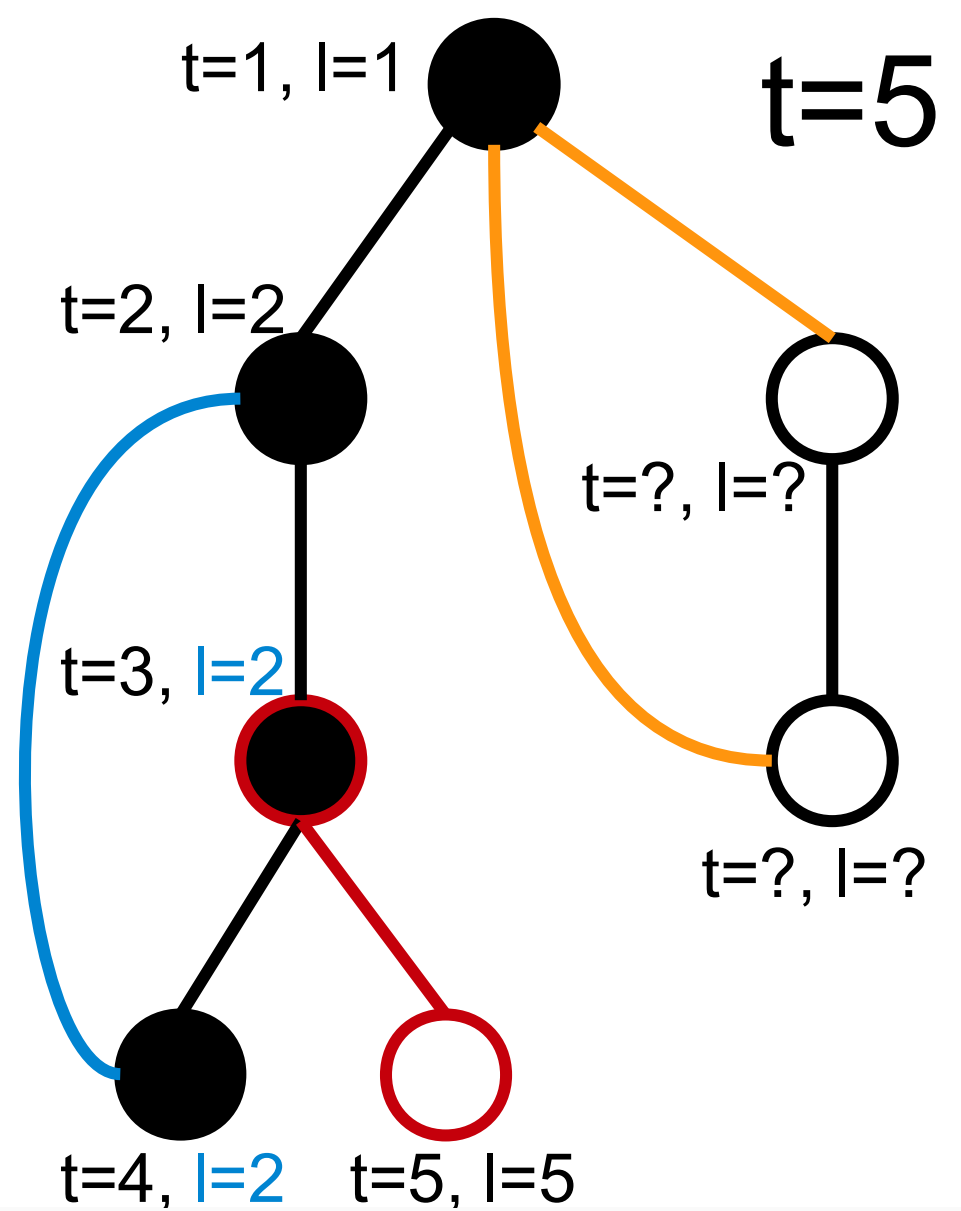

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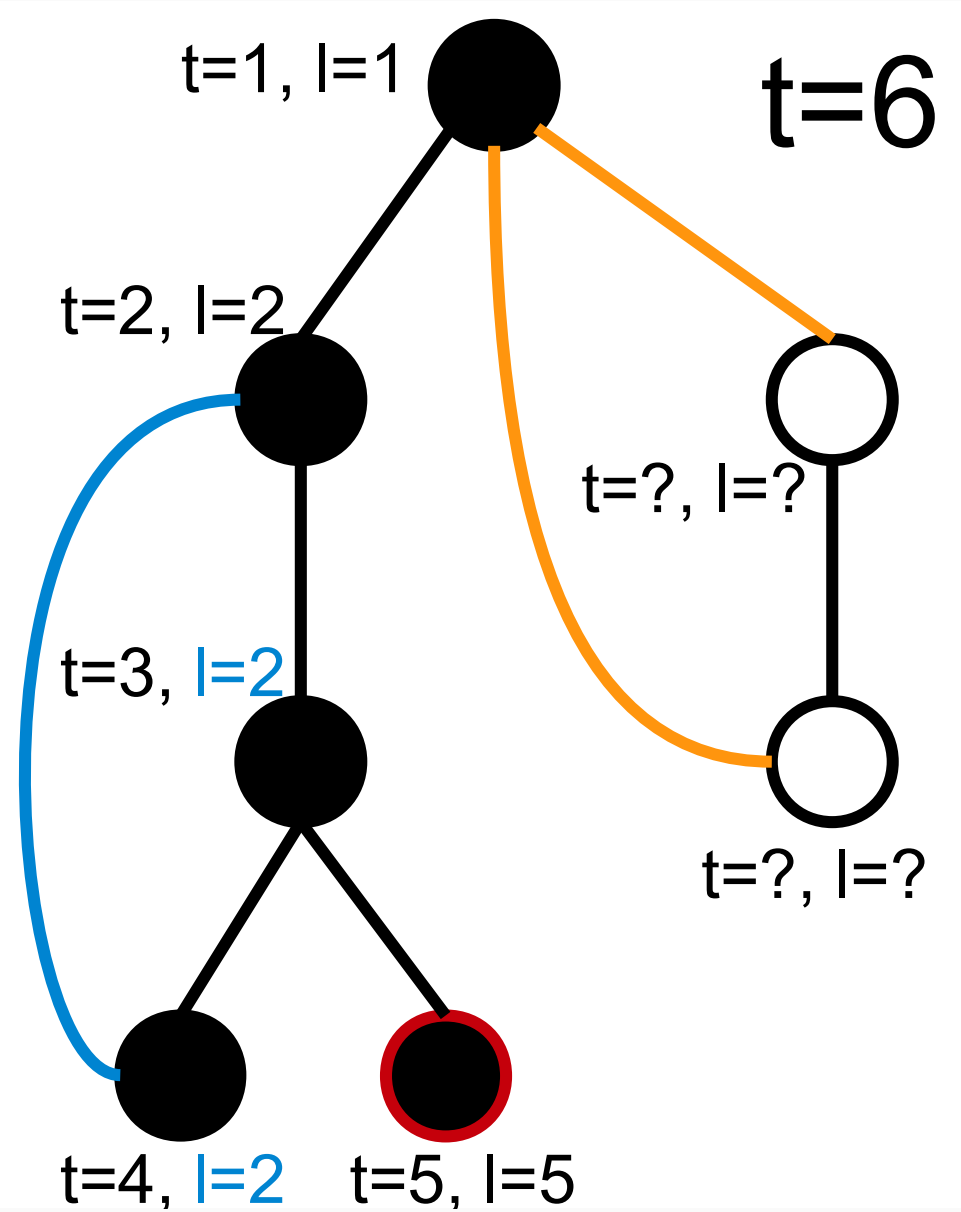

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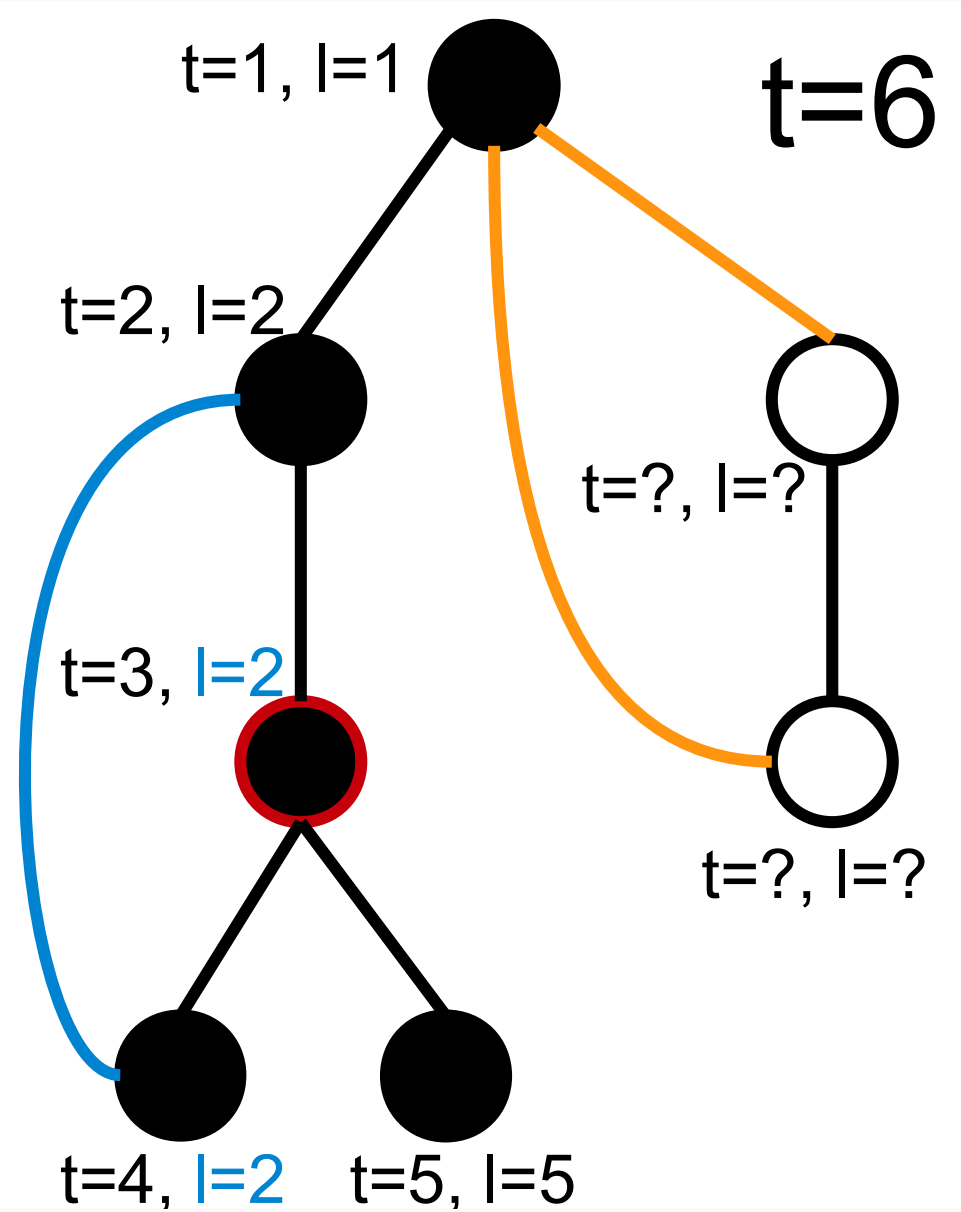

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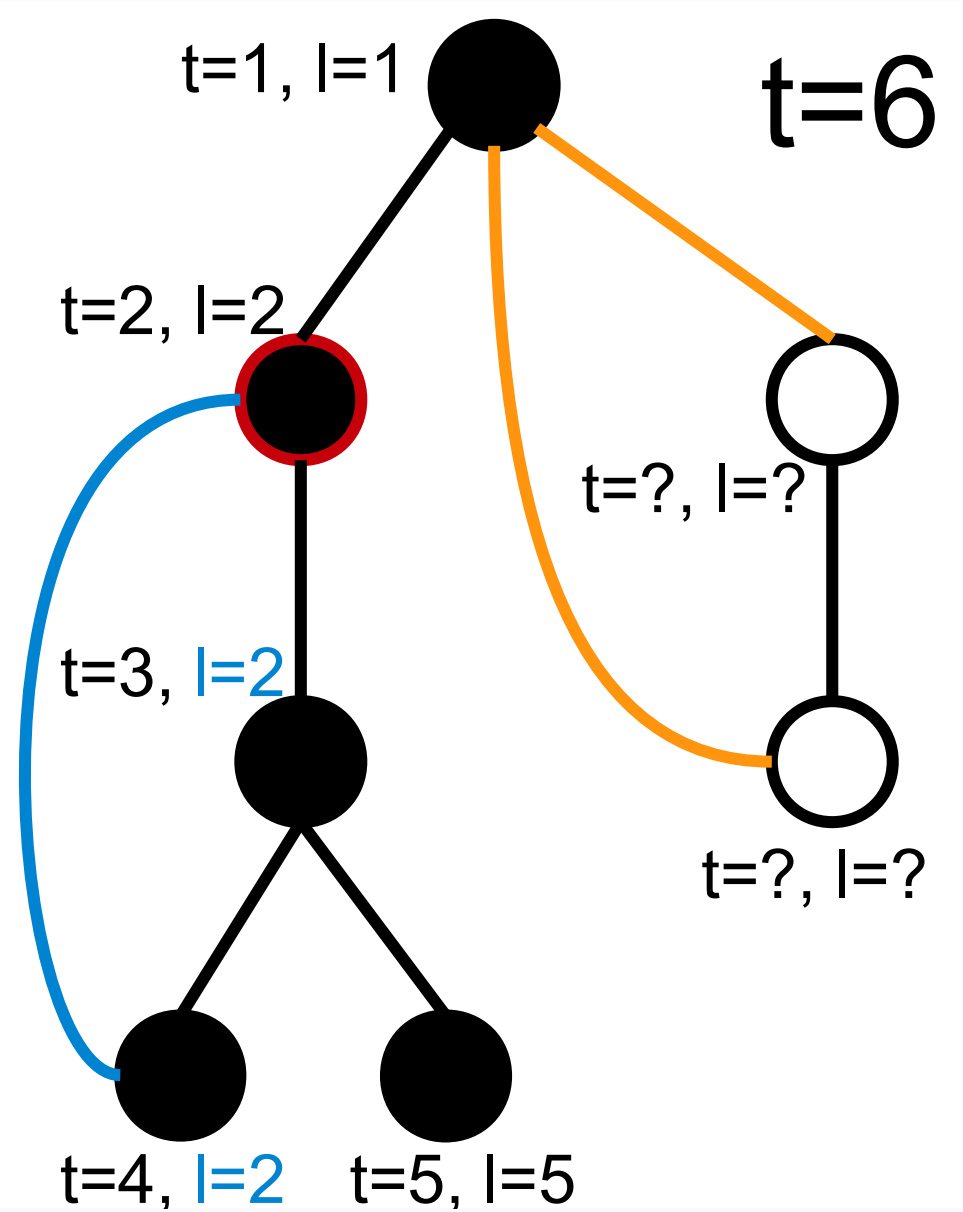

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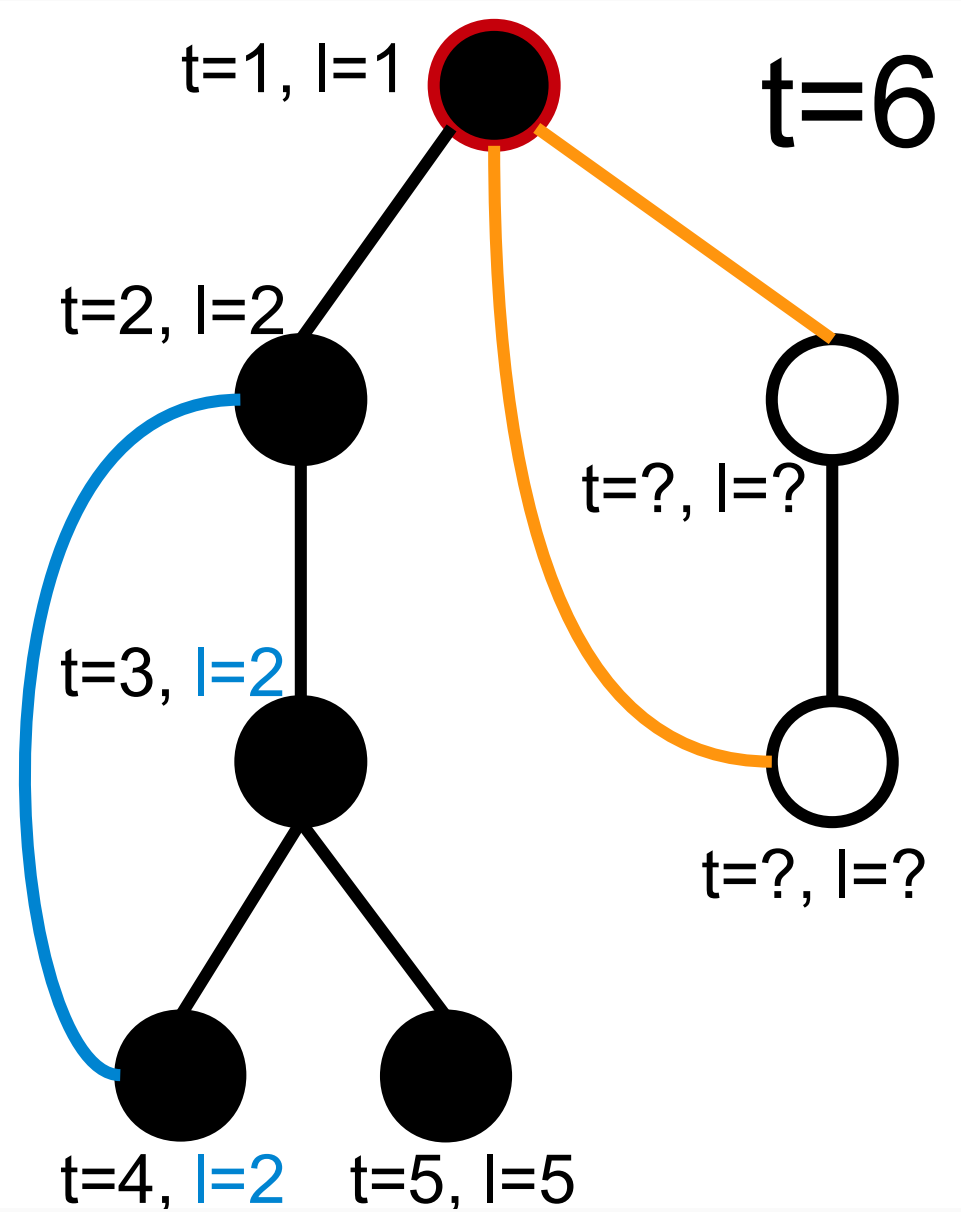

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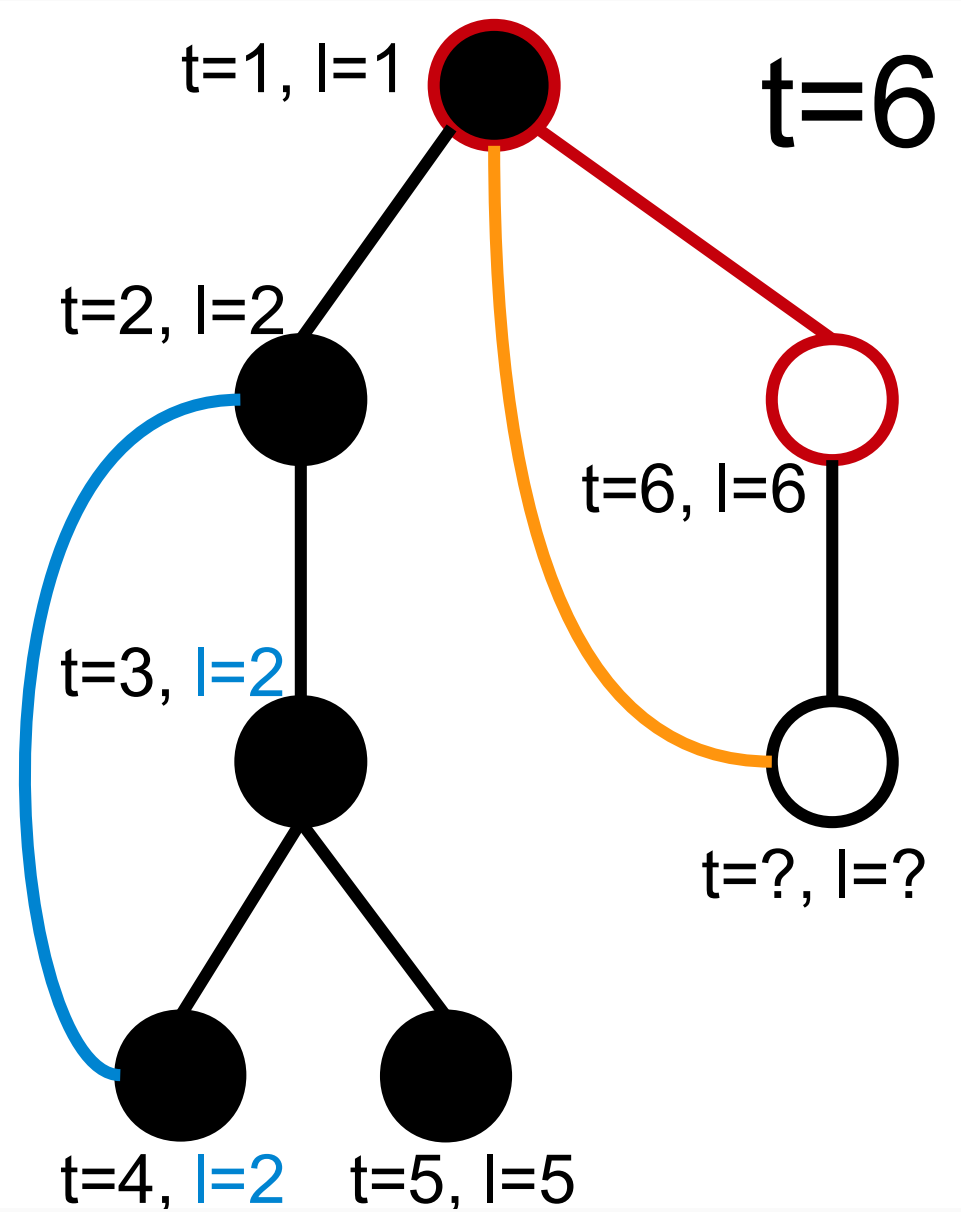

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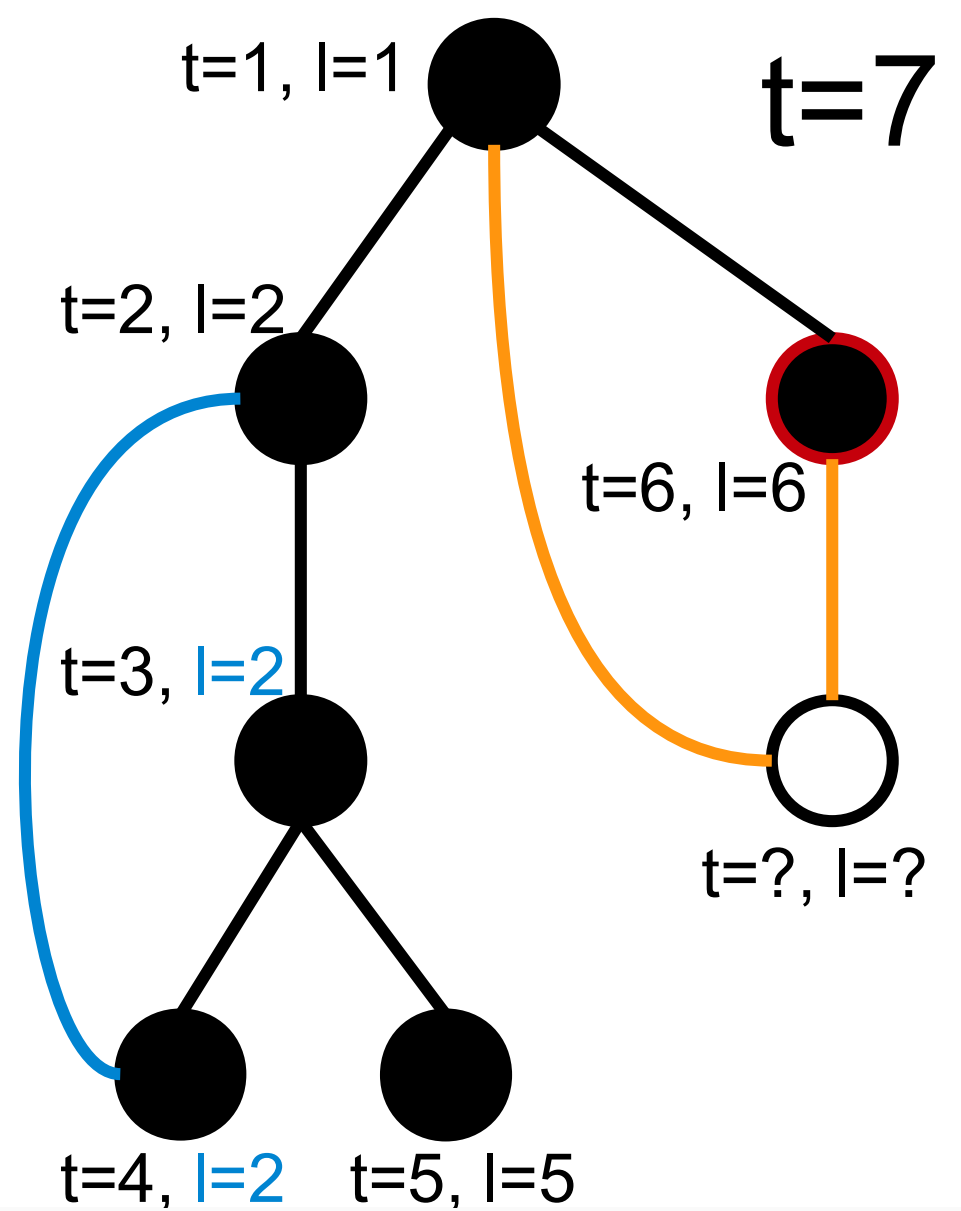

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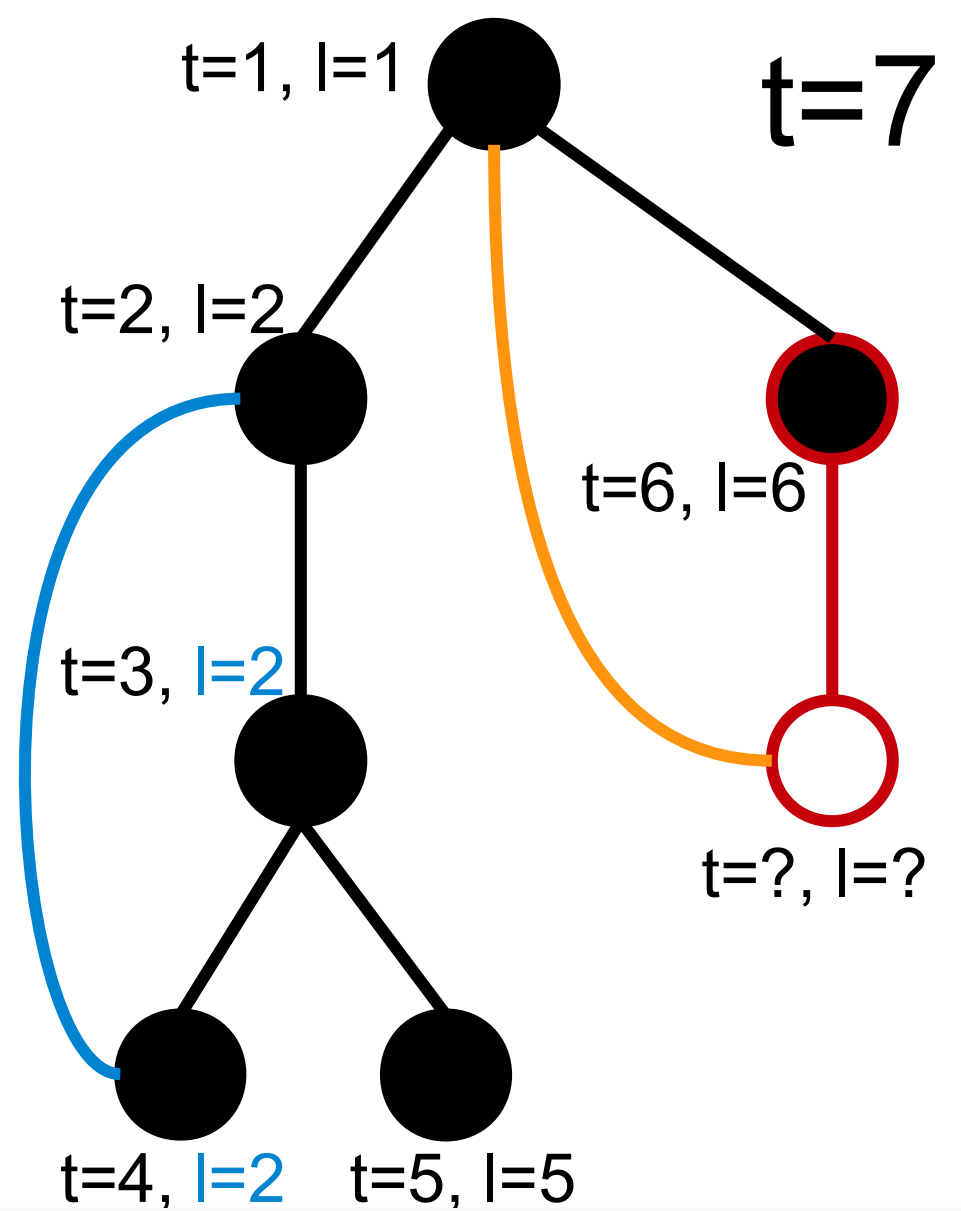

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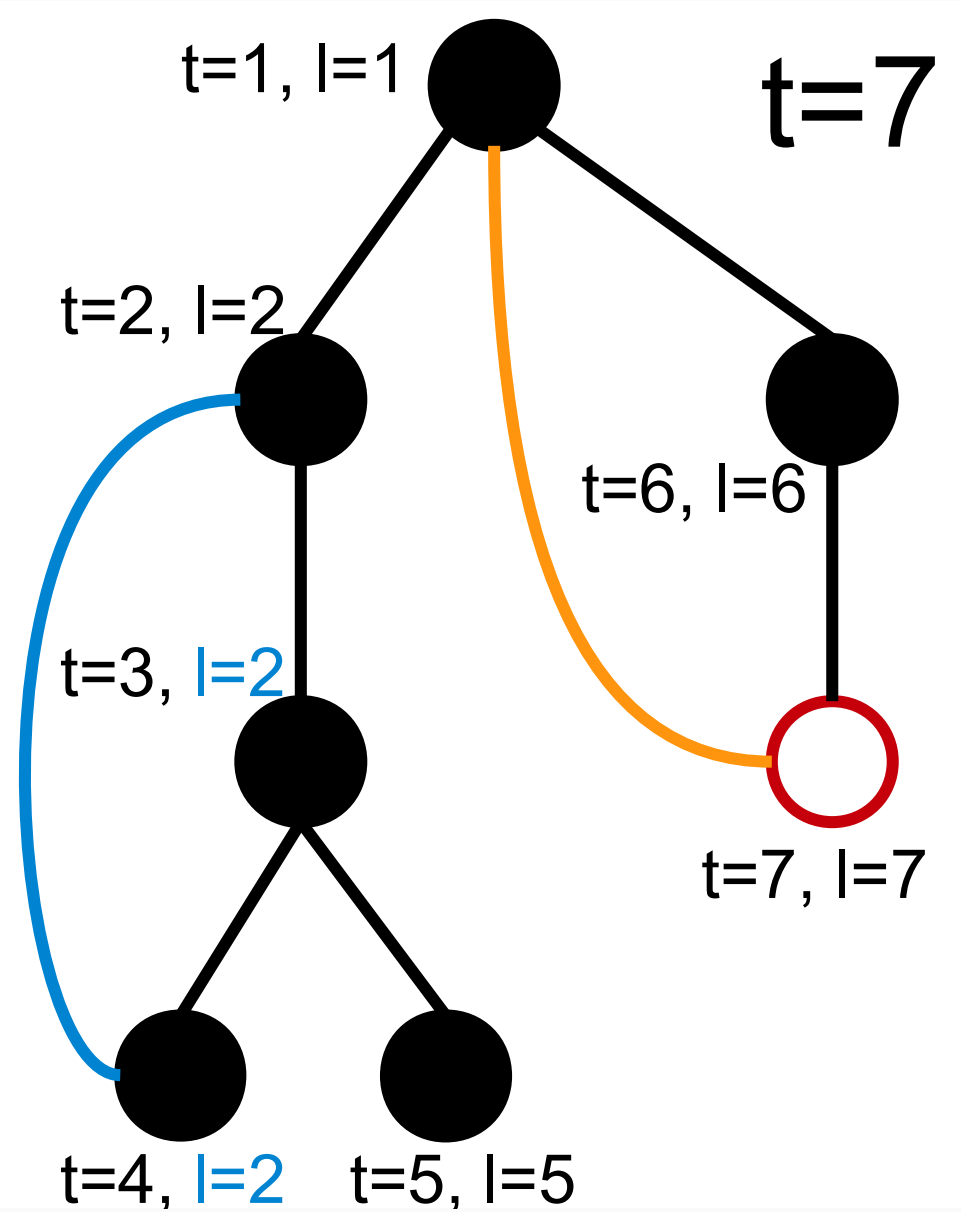

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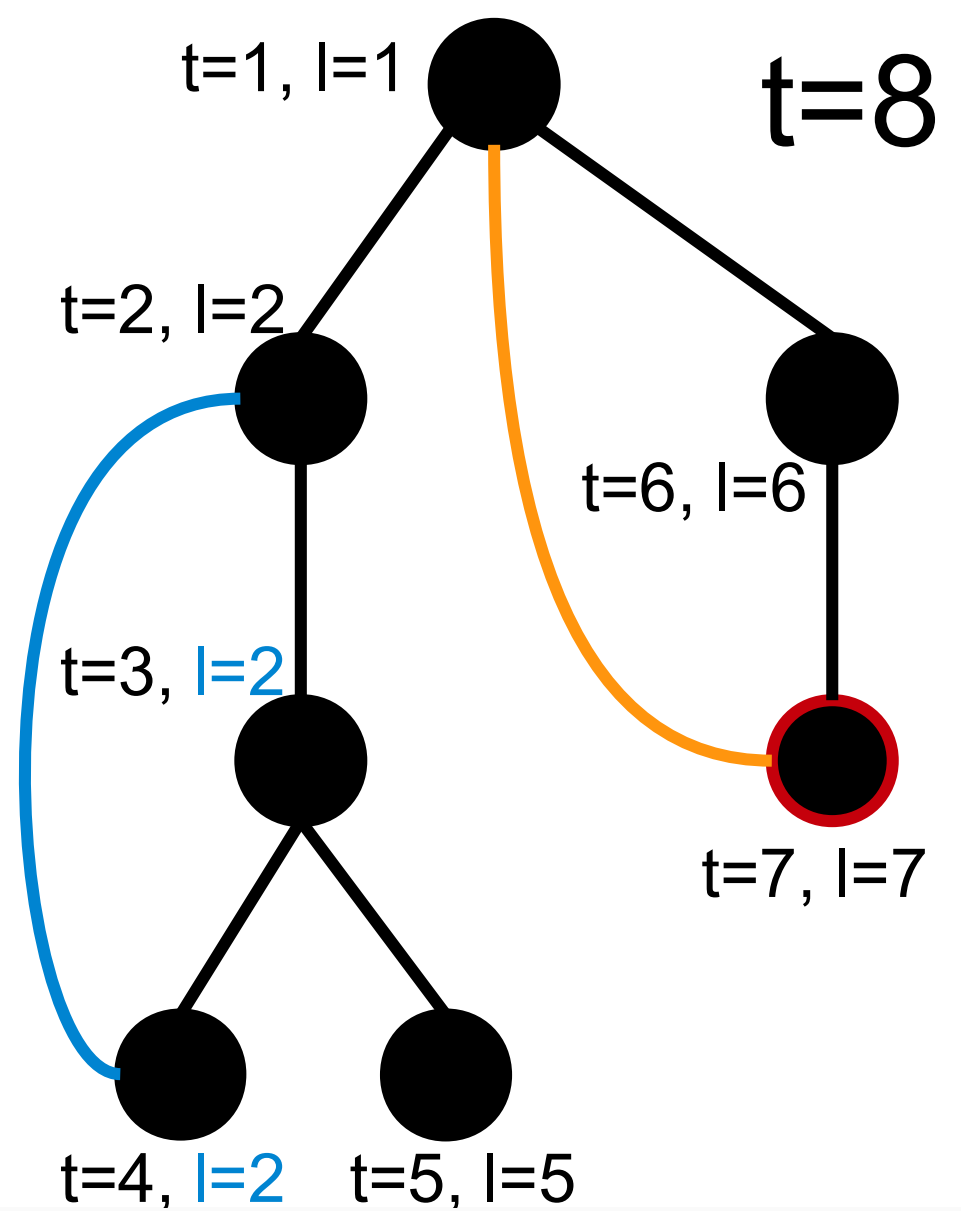

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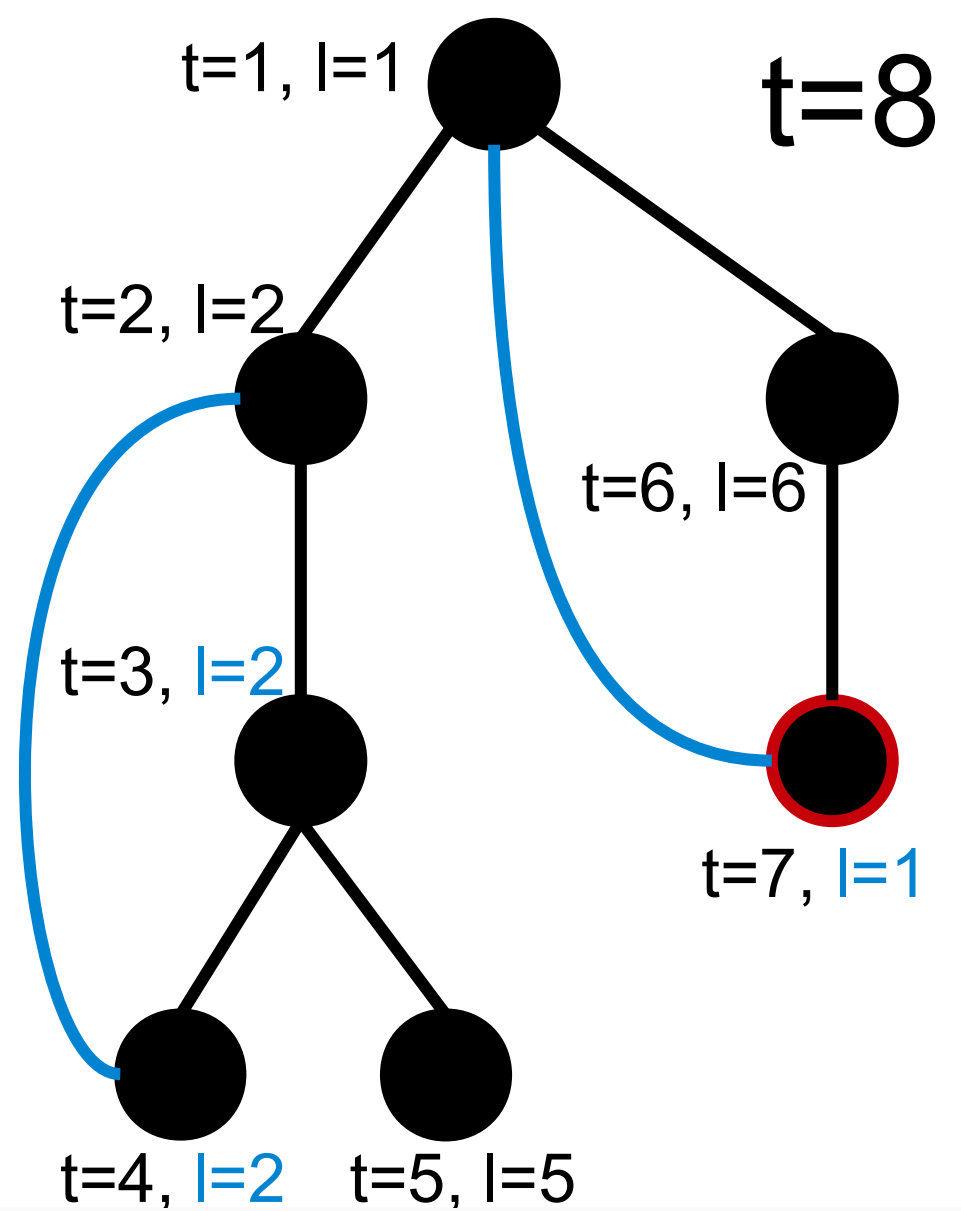

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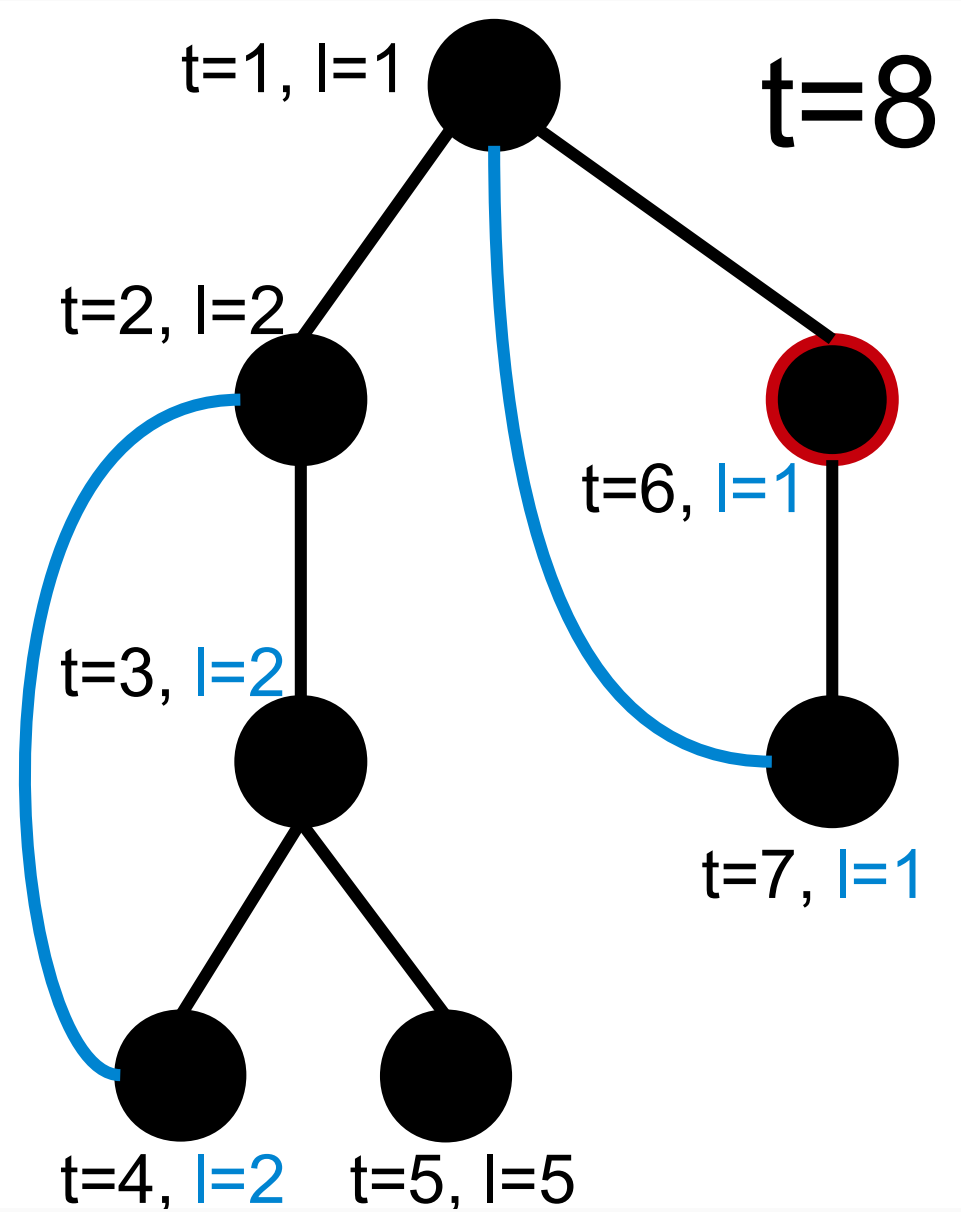

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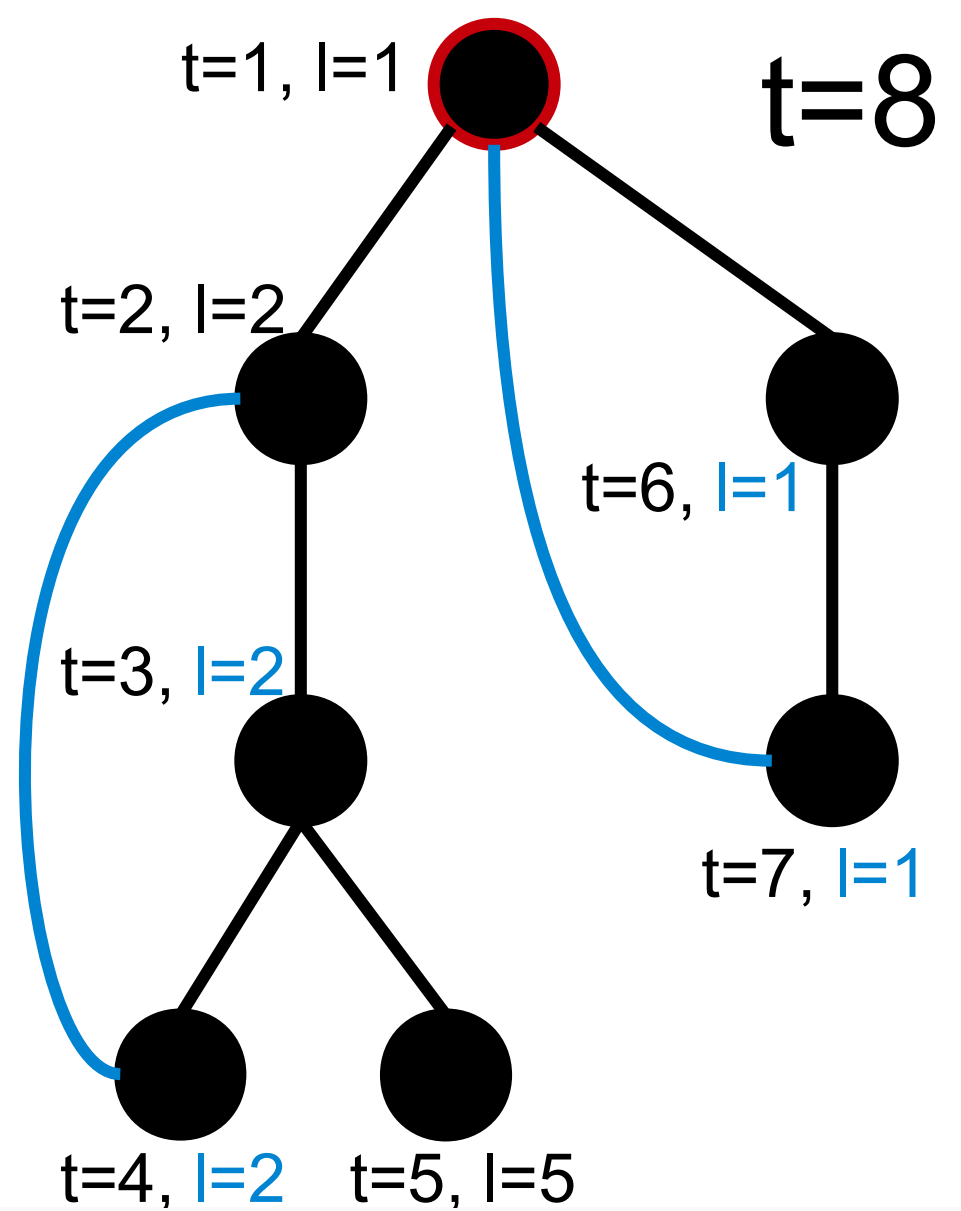

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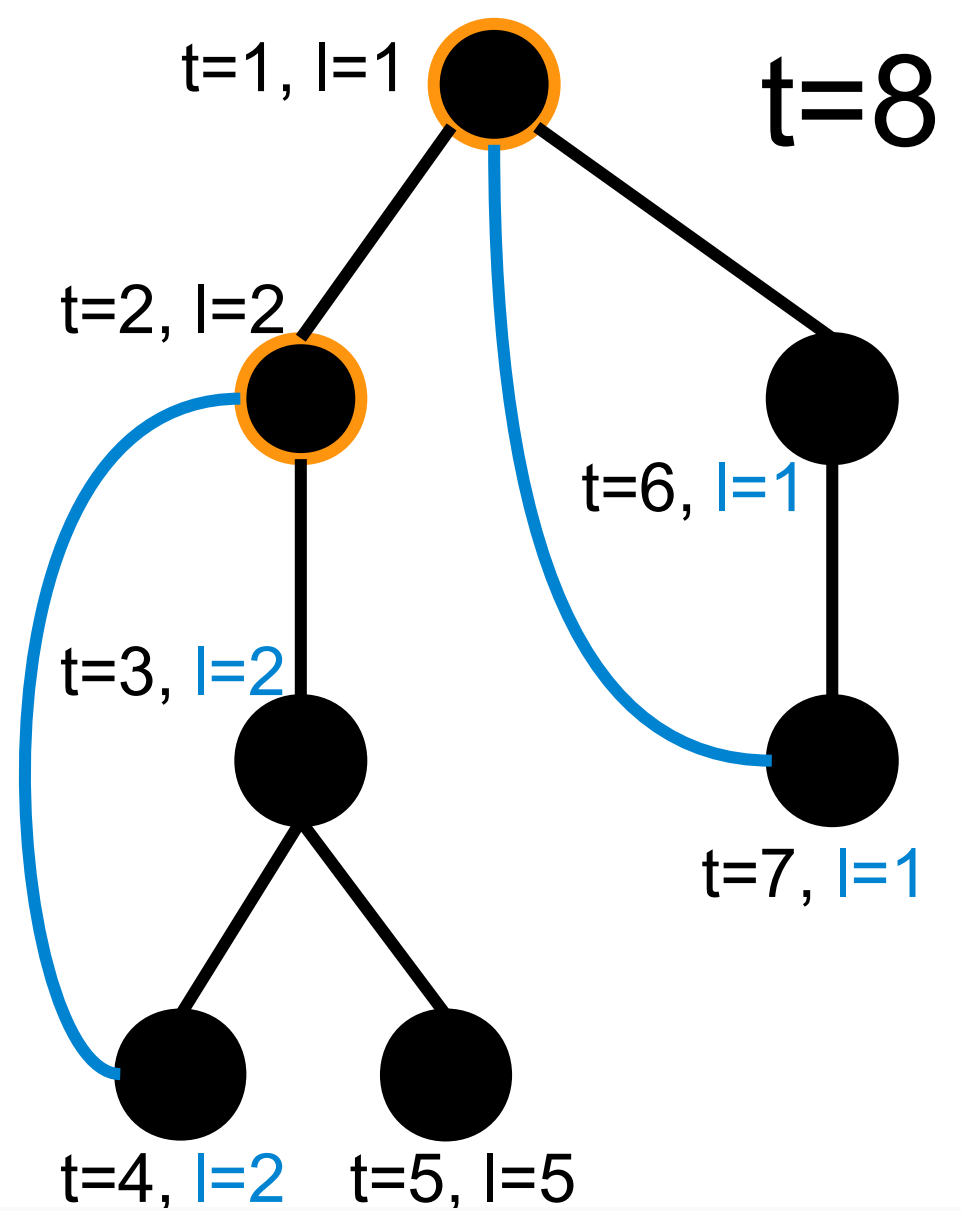


Ordering nodes

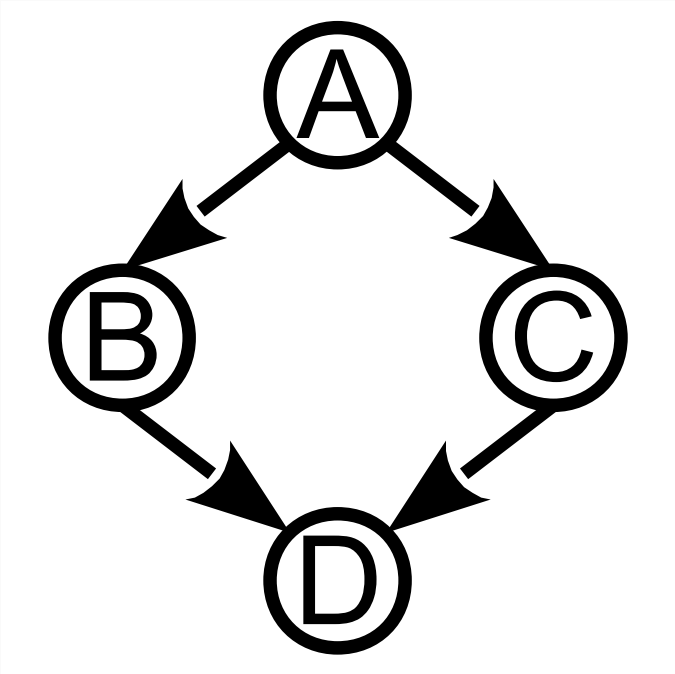
Types of Node Orderings

DFS $\rightarrow$ order **linearly** nodes in G.

- **Preordering:** Order of the 1st visit.
- **Postordering:** Order of the last visit.
- **Reverse Postordering:**
  Opposite order of the last visit $\rightarrow$ **topological sorting**

- **Topological sorting:**
  $\forall \langle u, v \rangle \in E$, u comes before v in the ordering.

Example



DFS Traversal (A, B, D, B, A, C, A) or (A, C, D, C, A, B, A).

- Preordering: (A, B, D, C) or (A, C, D, B)
- Postordering: (D, B, C, A) or (D, C, B, A)
- Reverse Postordering: (A, C, B, D) or (A, B, C, D)