Eigenvalue Decomposition

Diagonalization, Geometry, and the Spectral Theorem

Sergio Peignier

Linear Algebra / Applied Mathematics

Why Eigenvalue Decomposition?

Eigenvalue decomposition analyzes a linear transformation by expressing it in a basis where it becomes **simple**.

- Eigenvectors: principal directions of action
- Eigenvalues: scaling factors along those directions

Key idea: In the eigenvector basis, a complicated transformation becomes diagonal.

Geometric Interpretation

Along an eigenvector x:

$$Ax = \lambda x$$

- Direction is preserved
- Only stretched or contracted

Eigenvalue decomposition: change of coordinates revealing the matrix intrinsic geometry.

Applications of Eigenvalue Decomposition

Eigenvalue decomposition is fundamental in many areas:

- **Differential equations**: $\dot{x} = Ax$
- **Stability analysis**: sign and magnitude of eigenvalues
- **Network analysis**: steady states and ranking

Eigenvalue Decomposition

Let $A \in \mathbb{R}^{n \times n}$ be square and diagonalizable. Then:

$$A = XDX^{-1}$$

- D: diagonal matrix of eigenvalues $\lambda_1, \ldots, \lambda_n$
- X: matrix of eigenvectors

Each column x_i of X satisfies:

$$Ax_i = \lambda_i x_i$$

Action on a Vector

For any vector x:

$$Ax = XDX^{-1}x$$

$$Ax = X\left(D\left(X^{-1}x\right)\right)$$

Interpretation:

1. Transform x into eigenvector coordinates (X^{-1})
2. Stretch each coordinate by its eigenvalue (D)
3. Transform back to original space (X)

Solving Linear Systems

Consider:

$$Ax = b$$

Define:

$$\hat{x} = X^{-1}x, \qquad \hat{b} = X^{-1}b$$

Then:

$$D\hat{x} = \hat{b}$$

Decoupling into Scalar Problems

The system becomes:

$$\lambda_i \hat{x}_i = \hat{b}_i, \qquad i = 1, \ldots, n$$

- $n \times n$ system
- $\Rightarrow n$ independent scalar equations

This explains the computational power of diagonalization.

When Is a Matrix Diagonalizable?

Two necessary conditions:

- The matrix must be **square**
- It must have n linearly independent eigenvectors

Algebraic vs Geometric Multiplicity

For an eigenvalue λ:

- **Algebraic multiplicity (AM)**: number of times λ appears as a root of

$$p_A(x) = \det(xI - A)$$

- **Geometric multiplicity (GM)**:

$$\mathrm{GM}(\lambda) = \dim \ker(A - \lambda I)$$

Diagonalizability Criterion

A matrix is diagonalizable if and only if:

$$\mathrm{GM}(\lambda) = \mathrm{AM}(\lambda) \quad \text{for all eigenvalues } \lambda.$$

If $\mathrm{GM} < \mathrm{AM}$:

- Not enough eigenvectors
- Matrix is **defective**

Practical Criteria

A matrix is diagonalizable if:

1. All eigenvalues are distinct
2. The matrix is symmetric (or Hermitian)
3. The numerically computed eigenvector matrix has full rank

Why Some Matrices Are Not Diagonalizable

Diagonalization requires stretching along n independent directions.
If directions collapse into the same eigenspace:

- Degrees of freedom are lost
- Diagonalization becomes impossible

Defective Matrix Example

Shear matrix:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

- Characteristic polynomial: $(x-1)^2$
- $AM = 2$
- Only one eigenvector $\Rightarrow GM = 1$

Thus, A is not diagonalizable.

Spectral Theorem: Intuition

Symmetric matrices:

- Do not twist or shear space
- Only stretch along orthogonal directions

Geometric picture:

$$\text{rotate} \rightarrow \text{stretch} \rightarrow \text{rotate back}$$

Spectral Theorem

Theorem (Spectral Theorem)

If $A = A^T$ (real symmetric), then:

$$A = Q\Lambda Q^T$$

where:

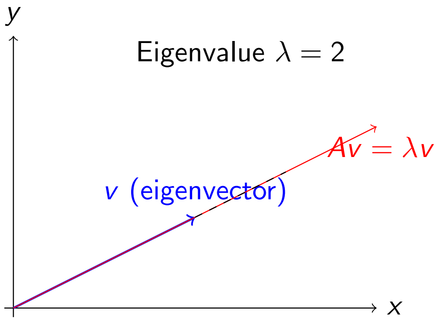
- Λ: *real eigenvalues*
- Q: *orthogonal matrix of eigenvectors*

Why It Works: Key Properties

For symmetric (Hermitian) matrices:

- Eigenvalues are real
- Eigenvectors of distinct eigenvalues are orthogonal
- An orthonormal eigenbasis exists

Visualizing a Real Eigenvalue and Eigenvector



- For a real eigenvalue λ, the eigenvector v is stretched or shrunk along the same line
- Direction does not change
- Scaling factor $= \lambda$

Worked Example

Let:

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$$

- Eigenvalues: $\lambda_1 = 3,\ \lambda_2 = 2$
- Eigenvectors:

$$x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Diagonalization of the Example

$$X = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \qquad D = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

Solving $Ax = b$ reduces to:

$$D\hat{x} = \hat{b}$$

with independent equations:

$$3\hat{x}_1 = \hat{b}_1, \qquad 2\hat{x}_2 = \hat{b}_2$$

Summary

Key Takeaways

- Eigenvectors define invariant directions
- Eigenvalues define stretching or contraction
- Diagonalization decouples linear systems
- A matrix must be square and diagonalizable
- Symmetric matrices are always diagonalizable

Thank You

Questions or discussion?

Complex Eigenvalues and Eigenvectors: Motivation

- Eigenvalues and eigenvectors describe how a linear map acts on vectors
- For real matrices, eigenvalues are not always real
- What does a **complex eigenvalue** mean geometrically?

Eigenvalue Definition (Review)

A scalar λ and nonzero vector v satisfy

$$Av = \lambda v$$

- λ is an eigenvalue
- v is an eigenvector

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Question: What happens if λ is complex?

Key Fact for Real Matrices

Let A be a **real matrix**.

Fact

If λ is a non-real (complex) eigenvalue of A, then the corresponding eigenvector must be complex.

Why the Eigenvector Cannot Be Real

Assume (for contradiction):

- A has real entries
- $\lambda \in \mathbb{C} \setminus \mathbb{R}$
- v is a **real** eigenvector

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but

$$\lambda v \quad \text{is complex}$$

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Contradiction

A real vector cannot equal a complex vector.

Structure of a Complex Eigenvector

Let

$$v = a + ib, \quad a, b \in \mathbb{R}^n$$

- a = real part
- b = imaginary part
- v itself is not geometrically real

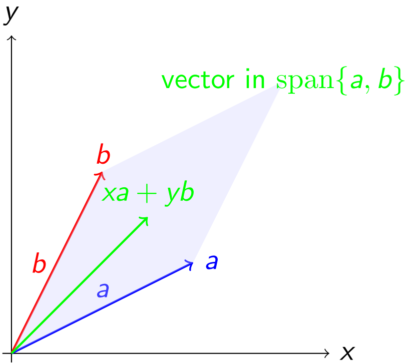
The Real Geometric Object

Although v is complex:

- a and b are real vectors
- They span a real 2D subspace

Invariant subspace: $\mathrm{span}\{a, b\}$

Visualizing the Span



- Any linear combination $xa + yb$ lies in the shaded plane
- This plane is the real 2D invariant subspace of the complex eigenvector

Action of the Matrix on This Plane

Let

$$\lambda = re^{i\theta}$$

Then on the plane $\mathrm{span}\{a, b\}$:

- scaling by factor r
- rotation by angle θ

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Matrix form in basis $\{a, b\}$:

$$r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Geometric Interpretation

- No real eigenvectors in the plane
- Vectors rotate and scale
- Leads to spirals or circles in dynamical systems

Key Takeaways

- Complex eigenvalue ⇒ complex eigenvector
- Real and imaginary parts span a real invariant plane
- The matrix acts as rotation + scaling on that plane
- *Complex eigenvalues of real matrices represent rotation and scaling in a real two-dimensional invariant subspace.*