Linear Systems: From Equations to Matrices

What Is a Linear System?

Definition

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- Linear systems arise from:
  - experimental measurements,
  - physical constraints,
  - biological interactions,
  - statistical models.
- They are central to:
  - regression,
  - numerical simulation,
  - PCA and matrix factorization.

A Simple Motivating Example (Biology)

Biological setting

A metabolite is regulated by two genes:

- gene a with regulatory strength x,
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For gene expression levels (E_a, E_b):

$$M = xE_a + yE_b$$

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Two experiments

$$\begin{cases} 2x - y = 3 \\ x + 4y = 10 \end{cases}$$

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- Each experiment provides a **linear constraint**
- The unknowns (x, y) must satisfy *all* constraints simultaneously

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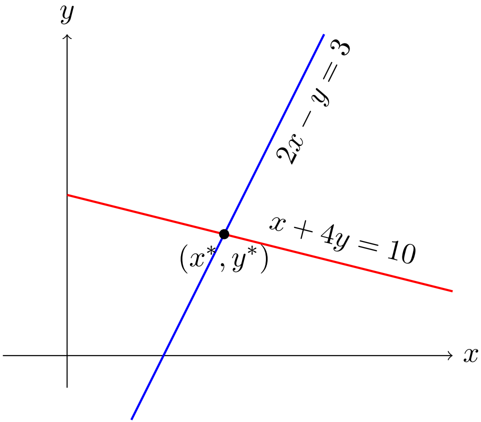
Goal

Find the pair (x, y) that explains all experimental observations.

Geometric Interpretation

- Each equation defines a straight line in the plane
- The solution is the intersection of the lines



From Equations to Matrices

- Coefficients $\rightarrow$ matrix
- Unknowns $\rightarrow$ vector
- Observations $\rightarrow$ vector

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$$A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix}, \quad x = \begin{pmatrix} x \\ y \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 10 \end{pmatrix}$$

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Compact form

$$\boxed{Ax = b}$$

Why Matrix Decompositions?

- Direct inversion is often:
  - numerically unstable,
  - computationally expensive.
- Modern solvers use **matrix factorizations**

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Two key decompositions

- LU decomposition
- QR decomposition

LU Decomposition

Factorization

$$A = LU$$

- L: lower triangular (unit diagonal)
- U: upper triangular

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Solving $Ax = b$

$$LUx = b$$

1. Forward substitution: $Ly = b$
2. Backward substitution: $Ux = y$

QR Decomposition

Factorization

$$A = QR$$

- Q: orthogonal matrix ($Q^\top Q = I$)
- R: upper triangular

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Solving

$$QRx = b \Rightarrow Rx = Q^\top b$$

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Why QR?

Orthogonal matrices preserve lengths:

$$\|Qx\| = \|x\|$$

Why QR Is Important

- More numerically stable than LU
- Handles:
  - ill-conditioned systems,
  - overdetermined systems ($m > n$),
  - least-squares regression.
- Central in:
  - regression,
  - eigenvalue algorithms.

Summary

Key ideas

- Linear systems model multiple constraints simultaneously
- Matrix form: $Ax = b$
- Geometry: intersection of hyperplanes

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Numerical solutions

- LU: efficient for square systems
- QR: stable and essential for least squares