Linear Algebra and Matrix Analysis

Space Vector

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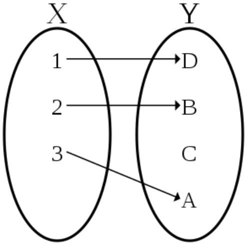
Field (corps commutatif) | Recall

Set K provided with operations of **addition** $(+)$ $(K \times K \to K)$ **and multiplication** $\cdot$ $(K \times K \to K)$, satisfying the field axioms ($\forall a, b, c \in K$):

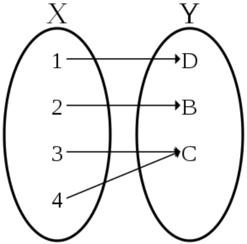
- Addition associativity: $a + (b + c) = (a + b) + c$
- Multiplication assortativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
- Addition commutativity: $a + b = b + a$
- Multiplication commutativity: $a \cdot b = b \cdot a$
- Additive identity: $a + 0 = a$
- Multiplicative identity: $a \cdot 1 = a$
- Additive inverse: $a + (-a) = 0$
- Multiplicative inverse: $a \cdot a^{-1} = 1, \quad \forall a \neq 0$
- Distributivity of multiplication over addition: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

Injection, Bijection, Surjection

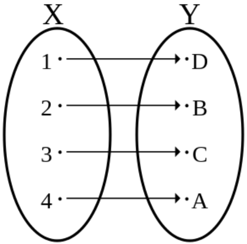
X: Domain, Y: Co-domain



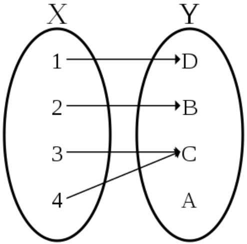
Injection



Surjection



Bijection



General case

Support

Subset of the domain containing the elements that are not mapped to zero.

$$Supp(f) = \{x \in X \quad \text{s.t.} \quad f(x) \neq 0\}.$$

Finite support

Vector Space | Definition

A **Set** E is a **Vector Space** over a **Field** K (e.g., $\mathbb{R}$ if it has two **operations**: $+$ and $\cdot$ s.t.:

- **Addition** $+ : E \times E \to E$, $\forall u, v, w \in E$
  - **Associative**: $u + (v + w) = (u + v) + w$
  - **Commutative**: $u + v = v + u$
  - **Additive identity element** 0_E: $u + 0_E = a$ (zero vector)
  - **Additive inverse element**: $u + (-u) = 0$
- **Scalar multiplication** $\cdot : K \times E \to E$, $\forall k, k_1, k_2 \in K$
  - **Distributivity w.r.t** E: $k \cdot (u + v) = k \cdot u + k \cdot v$
  - **Distributivity w.r.t** K: $(k_1 + k_2) \cdot u = k_1 \cdot u + k_2 \cdot u$
  - **Associative**: $k_1 \cdot (k_2 \cdot a) = (k_1 \cdot k_2) \cdot a$
  - **Scalar multiplicative identity**: $1 \cdot v = v$

Examples

- K is a vector space over K.
- $\mathbb{C}$ is a vector space over $\mathbb{R}$.
- The set of continuous functions C^0 is a vector space over $\mathbb{R}$.

Exercise

Let $E = \mathbb{R}_+ = \{x \in \mathbb{R} | x > 0\}$, provided with the inner addition operator $\oplus$ defined as $a \oplus b = ab$, for all $a, b \in E$. And let the scalar multiplication law $\otimes$ defined as $\lambda \otimes a = a^{\lambda}$, for all $\lambda \in \mathbb{R}$ and all $a \in E$. Show that E provided with these two laws is a vector sapce over $\mathbb{R}$.

Subspace

E vector space over K. $\boxed{F \subseteq E}$ is a **subspace** of E if:

- $\boxed{F \neq \emptyset}$
- $\forall u, v \in F; \quad \boxed{u + v \in F}$
- $\forall k \in K, u \in F; \quad \boxed{k \cdot v \in F}$

Examples

- E is a subspace of E.
- $\{0_E\}$ is a subspace of E.
- $0_E \in F$ for all subspace $F \subseteq E$.
- If $x \in E$ and $x \neq 0_E$, then $\{\lambda \cdot x | \lambda \in K\}$ is a subpace.
- If F and G are subspaces of E then $F \cap G$ is a subspace of E.

Exercises

Which sets $E_1, ..., E_4$ are subspaces of $\mathbb{R}^3$. Compute their dimensionality.

1. $E_1 = \{(x, y, z) \in \mathbb{R}^3 | x + y - z = x + y + z = 0\}$.
2. $E_2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 - y^2 = 0\}$.
3. $E_3 = \{(x, y, z) \in \mathbb{R}^3 | e^x e^y = 0\}$.
4. $E_4 = \{(x, y, z) \in \mathbb{R}^3 | z(x^2 + y^2) = 0\}$.

Linear transformation

Let E and F be two vector spaces over K. A function $f : E \to F$ is a **Linear transformation** of:

$$f(x+y) = f(x) + f(y) \qquad \forall x, y \in E \tag{1}$$

$$f(\lambda x) = \lambda f(x) \qquad \forall x \in E, \lambda \in K \tag{2}$$

- **Isomorphism**: Bijective linear transformation.
- Inverse of an isomorphism is also a linear transformation.
- **Endomorphism**: Linear transformation from E to E.
- **automorphism**: Bijective endomorphism.

Kernel

E and F vector spaces over K, and $f : E \to F$ a linear mapping. the **Kernel** of f is the set:

$$\ker f = \{x \in E | f(x) = 0_E\} \tag{3}$$

$$0_E \in \ker f, \quad \text{i.e.,} \quad f(0_E) = 0_F \quad \forall f \tag{4}$$

Examples

- $id_E(x) = x \quad \forall x \in E$ is an automorphism
- If $f : E \to F$ is a linear mapping, then $x \to \lambda f(x)$ is also a linear mapping
- If $g : E \to F$ is a linear mapping, then $x \to f(x) + g(x)$ is a linear mapping (denoted $f + g$).
- $\mathcal{L}_K(E, F)$, the set of linear mappings from E to F is a vector space over K.

Exercice

- Prove the previous examples
- Give an example of surjective endomorphism.
- Give an example of injective endomorphism

Indexed Family

- E vector space over K
- I a set (index).

- **Indexed family**: $(x_i)_{i \in I}$
  mapping $i \to x_i$ from I to E

Subfamily

Family $(y_j)_{j\in J}$ is a **subfamily** of $(x_i)_{i\in I}$ if:

- $J \subset I$
- $y_j = x_j$ for all $j \in J$.

Free family

Family $(x_i)_{i\in I}$ is **free** if $\forall(\lambda_i)_{i\in I}$, $\lambda_i \in K$, with **finite support**:

$$\sum_{i\in I} \lambda_i x_i = 0 \quad \Longrightarrow \lambda_i = 0, \quad \forall i \in I \tag{5}$$

(linearly independent)

Generating family

Family $(x_i)_{i\in I}$ is a **generating set** if:
$\forall x \in E, \quad \exists(\lambda_i \in K)_{i\in I}, \quad$ with finite support, s.t.:

$$x = \sum_{i\in I} \lambda_i x_i. \tag{6}$$

Basis

Family $(x_i)_{i\in I}$ is a **base** of E if it is **free** and **generating**

- $(x_i)_{i\in I}$ are **linearly independent** and every $x \in E$ is a **linear combination** of $(x_i)_{i\in I}$.
- $\forall x \in E$: $\exists!(\lambda_i \in K)_{i\in I}$, with finite supp., s.t., $x = \sum_{i\in I} \lambda_i x_i$.
- λ_i est the **coordinate of** x in $(x_i)_{i\in I}$.

Equivalent Conditions

E vector space over K, $\mathcal{B} = (b_i)_{i \in \{1,\dots,n\}}$ family of E.
The following statements are equivalent:

i) $\mathcal{B}$ is a **basis**.

ii) $\mathcal{B}$ is a **minimal generating** set of E (No proper subfamily is generating)

iii) $\mathcal{B}$ is a **maximal set of linearly independent vectors** of E (no other linearly independent set contains it as a proper subset).

Incomplete Basis Theorem

E vector space over K.
$(g_i)_{I \in I}$ **finite generating set**.
$(\ell_j)_{j \in J}$ **linearly independent** finite **subfamily** of g.
Then ℓ is a **subfamily finite basis** $\mathcal{B}$ of E, **subfamily** of g.

A basis is the:

- Smallest generating set (if one element is removed → no longer generating)
- Largest linearly independent family (if one element x is added, → x is a linear combination of the basis)

Dimension

E v.s. over K, s.t., E has a **finite generating family**.

i) E has a basis $\mathcal{B}$ indexed by a **finite set** and the **cardinal** of this set (nb. of elements) is $\operatorname{card}\mathcal{B}$.

ii) If $\mathcal{B}'$ is another basis of E, $\operatorname{card}\mathcal{B}' = \operatorname{card}\mathcal{B}$. (dimension theorem)

$\operatorname{card}\mathcal{B}$ is the **dimension of** E **over** K, termed $\boxed{\dim_K E}$.

If $E = \{0\}$ **then**: $\dim_K E = 0$, .

Alternative Basis Definition

E v.s. over K, s.t. $\dim_K E = n$; $\mathcal{B}$ finite family of E. $\mathcal{B}$ is a basis of E if two properties of these properties are true.

i) $\operatorname{card} \mathcal{B} = n$.

ii) $\mathcal{B}$ is a **linearly independent** family E.

iii) $\mathcal{B}$ is a **generating set** of E.

Isomorphic Vector Space

E and F v.s. over K with finite basis. The following statements are equivalent:

i) E and F are isomorphic ($\exists$ an isomorphism from E to F).

ii) $\dim_K E = \dim_K F$.

Finite v.s., differing by an isomorphism, are classified by their dimension $n \in \mathbb{N}$.

Existence of a basis

Let E be a v.s. over K (not necessarily finite), and let $(e_i)_{i \in I}$ be a linearly independent family of E.
Then $(e_i)_{i \in I}$ is a subfamily of $(f_j)_{j \in J}$ which is linearly independent and generating.
Then E has a basis.

Dimension of linear applications

- E and F finite s.v. over K, s.t. $\dim_K E = n \geq 1$ and $\dim_K F = p \geq 1$, with basis $(e_i)_n$ et $(f_j)_p$ resp.
- $\mathcal{L}(E, F)$ the v.s. of linear applications from E to F

The family of applications:

$$(f_{ji})_{p,n} = \begin{cases} f_{ji}(e_i) = f_j \\ f_{ji}(e_{i' \neq i}) = 0 \end{cases}$$

Is a base of $\mathcal{L}(E, F)$, and $\dim_K \mathcal{L}(E, F) = \dim_K E \cdot \dim_K F$

Matrix representation

- E and F finite s.v. over K, s.t. $\dim_K E = n$ and $\dim_K F = p$, with basis $(e_i)_n$ et $(f_j)_p$ resp.
- $f : E \to F$ linear transformation s.t.:

$$f(e_i) = \sum_{j=1}^{p} a_{ji} f_j. \tag{7}$$

The **matrix of** f **in the basis** $(e_i)_n$ **and** $(f_j)_p$, $\mathrm{Mat}(f, (e_i)_n, (f_j)_p)$ is the family $(a_{ji} \in K)_{(j,i)\in[1,p]\times[1,n]}$.

$$\mathrm{Mat}(f, (e_i)_n, (f_j)_p) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{p1} & a_{p2} & \dots & a_{pn} \end{pmatrix} \tag{8}$$

Matrix Vector Space

Linear mapping on a finite v.s. $\rightarrow$ **matrix** representation.
Let $M_{p,n}(K)$ ($M_n(K)$ if $p = n$) be the set of $n \times p$ **matrices** with values in K.
Providing **addition** and **multiplication by a scalar** s.t.:

- $[a_{ji}] + [b_{ji}] = [a_{ji} + b_{ji}]$.
- $\lambda[a_{ji}] = [\lambda a_{ji}]$, for $\lambda \in K$

Then $M_{p,n}(K)$ is a **vector space** over K.

Matrix Multiplication

Matrices $A \in M_{p,n}(K)$ and $B \in M_{n,\ell}(K)$
Their **matrix multiplication** $AB = C \in M_{p,\ell}(K)$ is s.t.:

$$c_{jk} = \sum_{i=1}^{n} a_{ji} b_{ik}. \tag{9}$$

Vectors

Vector: **matrix with 1 columns** $x \in M_{n,1}(K)$.
If $M \in M_{p,n}(K)$, then the product $Mx \in M_{p,1}$ is a vector.
The transformation $x \to Mx$ is a linear mapping from $M_{n,1}(K)$ to $M_{p,1}(K)$.
$M_{n,1}(K)$ and $M_{p,1}(K)$ are isomorph to K^n K^p, and the mapping is denoted as $\hat{M} : K^n \to K^p$.

Summary

In finite dimensions:

- **Linear mappings : matrices**
- **Elements of the v.s.: vectors**
- **Transformation** of an element: **product of a matrix by a vector**.

Rank Theorem

E and F v.s; and $f : E \to F$ linear transformation.
Suppose that E has a finite dimension.
Then: Im $f \equiv f(E)$ has a finite dimension s.t.

$$\dim_K(\mathrm{Im}\ f) + \dim_K(\ker f) = \dim_K E$$

The dimension of $f(E)$, is called the **rank of** f and is noted Rank f.

Corollary

E and F, v.s. over K s.t. $\dim_K E$ et $\dim_K F$ finite, and $f : E \to F$ linear application, then:

- Rank $f = \dim_K E$ iff f is **injective**.
- Rank $f = \dim_K F$ iff f is **surjective**.
- If $\dim_K E = \dim_K F$, the following properties are equivalent:
    i) f is **injective**.
    ii) f is **surjective**.
    iii) f is **bijective**.

Transpose

Let $A \in M_{p,n}(K)$ is a matrix $p \times n$. The **transpose** of A, called ${}^tA$ or $\bar{A}$, is a matrix $[b_{ij}]_{ij} \in M_{n,p}(K)$, s.t.: $b_{ij} = a_{ji}$.

- ${}^t(A+B) = {}^tA + {}^tB$, ${}^t(\lambda A) = \lambda {}^tA$, ${}^t(AC) = {}^tC {}^tA$
- ${}^t({}^tA) = A$
- A square matrix $A \in M_n K$ is **symmetric** if ${}^tA = A$.
- Rank $A =$ Rank ${}^tA$

Definition

$A = [a_{ij}]_{ij} \in M_n(K)$; a $n \times n$ Matrix.
Determinant of A: $\det A$ is the scalar:

$$\det A = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = \sum_{\sigma \in \text{perm}(n)} \text{sgn}(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} \dots a_{\sigma(n)n} \tag{10}$$

Where the **signature** of a permutation σ is

$$\text{sgn}(\sigma) = \begin{cases} 1; & \text{for } \sigma \text{ even (prod. of even nb. of transpositions)} \\ -1; & \text{For } \sigma \text{ odd (prod. of odd nb. of transpositions)} \end{cases}$$

Properties

1. $\det I = 1.$
2. $\det A = \det {}^t A.$
3. $\det[C_1|...|\lambda C_j|...|C_n] = \lambda \det C$
   where C_i are columns of C (idem for rows).
4. $\det[C_{\sigma(1)}...C_{\sigma(n)}] = \text{sgn}(\sigma) \det C$ (idem for rows).
5. If $\exists C_i = \sum_{j \neq i} \lambda_j C_j$ then $\det C = 0$ (idem for rows).
6. $\det[C_1| \dots |aC_i' + C_i''| \dots |C_n] = a \det[C_1| \dots |C_i'| \dots , C_n] + \det[C_1| \dots |C_i''| \dots |C_n$
7. $\det[C_{\sigma(1)}|...|C_{\sigma(i)} + \sum_{j \neq i} \lambda_j C_j|...|C_{\sigma(n)}] = detC$
8. A s.t. $a_{i1} = 0$ for $i \geq 2$ and $B = [b_{ij}]$ s.t. $b_{ij} = a_{i+1,j+1}$; then $\det A = a_{11} \det B$.
9. $\det AB = \det BA = \det A \det B$.
10. A invertible iff $\det A \neq 0$.

Laplace's formula

Expresses the determinant of a matrix in terms of its **minors**.
Let $A = [a_{ij}]_{ij} \in M_(K)$. then:

$$\det A = \sum_{k=1}^{n} (-1)^{k+j} a_{kj} \det A_{\neq k, \neq j} \tag{11}$$

$$= \sum_{k=1}^{n} (-1)^{i+k} a_{ik} \det A_{\neq i, \neq k}. \tag{12}$$

Corollary

Let $A \in M_{(}K)$ be a block triangular matrix:

$$A = \begin{bmatrix} A_1 & & & \\ 0 & A_2 & & \\ 0 & 0 & ... & \\ 0 & 0 & 0 & A_r \end{bmatrix}. \tag{13}$$

Then $\det A = \prod_{i=1}^{r} \det A_i$.