Linear Systems and Linear Regression

From Linear Systems to Regression

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Key idea

Linear regression is the extension of linear systems to data with noise and many observations.

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We represent each sample as a feature vector:

$$x_i = (\mathrm{TF1}_i,\ \mathrm{TF2}_i,\ \mathrm{ATP}_i)$$

Linear Regression Model

We assume that gene expression depends *approximately linearly* on the features:

$$y_i \approx c_1\,\mathsf{TF1}_i + c_2\,\mathsf{TF2}_i + c_3\,\mathsf{ATP}_i + c_4$$

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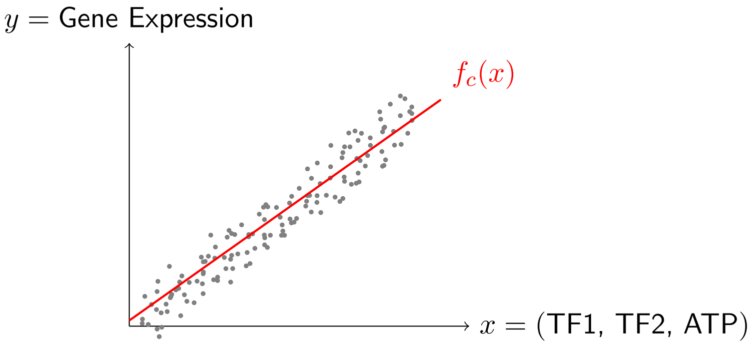
Interpretation

Linear regression estimates how much each factor contributes to the output.

Schematic View of Linear Regression



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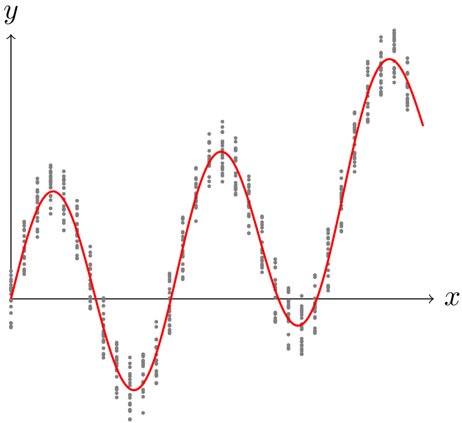
- $\sin(x)$ and x^2 are nonlinear
- coefficients c_1, c_2 enter linearly

Example: Nonlinear Features, Linear Regression



Problem Setup

We consider:

- m observations (x_i, y_i)
- each input vector:

$$x_i = (x_{i1}, \dots, x_{ip}) \in \mathbb{R}^p$$

- output $y_i \in \mathbb{R}$

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Goal

Find coefficients that best predict y_i from x_i.

Feature Transformations

Features can be arbitrary transformations:

$$\phi_1(x_i),\ \phi_2(x_i),\ \ldots,\ \phi_n(x_i)$$

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- polynomials: $x_{i1}^2, x_{i1}x_{i2}$
- nonlinear: $\log(x_{i2}+1), \sin(x_{i1})$

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Key message

Nonlinear behavior can be modeled with linear regression by choosing suitable features.

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Model:

$$f_c(x_i) = \sum_{j=1}^{n} c_j \, \phi_j(x_i)$$

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Least squares criterion

$$SSE = \sum_{i=1}^{m} (y_i - f_c(x_i))^2$$

Matrix Formulation

Design matrix:

$$A = \begin{pmatrix} \phi_1(x_1) & \cdots & \phi_n(x_1) \\ \vdots & \ddots & \vdots \\ \phi_1(x_m) & \cdots & \phi_n(x_m) \end{pmatrix}$$

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Compact form

$$\boxed{Ac \approx y}$$

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- Each row of A corresponds to one observation
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Special cases:

- $Ac = y$: exact fit
- $m > n$: overdetermined (least squares)
- $m < n$: underdetermined (many solutions)

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- Linear regression generalizes linear systems to data
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Matrix view

Regression reduces to solving

$$Ac \approx y$$

which connects directly to numerical linear algebra.