Stochastic Process

Arrival, Bernoulli and Poisson

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Stochastic process

Stochastic process | Definitions

- **Stochastic/random process** or **random function**.
- $\{X_t\}_{t\in T}$: **Collection of rvs** X_t **labeled** by an **index set** T (e.g., integers, reals), usually representing **time**.
- **Stochastic process** $\rightarrow$ 1D.
  Random field $\rightarrow$ more dimensions.
- **Sample function** or **realization**
  $\rightarrow$ **Outcome** of a **stochastic process** :
    $\rightarrow$ **Collection of its rvs' outcomes**.
- **Increment**: **rv** $X_{t_2} - X_{t_1}$ for $t_1 \in T$ and $t_2 \in T$, s.t. $t_2 \geq t_1$.

Taxonomy of random processes

Categories of index sets

- Discrete-time.
- Continuous-time.

Categories of state spaces (i.e., sample space of rvs)

- Discrete or integer-valued.
- Real-valued or continuous state space
- N-dimensional vector process

Examples

- Bernoulli process
- Poisson process
- Markov processes
- Markov chains
- Moran process
- Branching process
- Wiener process
- Random walks
- Lévy process
- Martingale
- ...

Stationarity

- $\{X\}_{t \in T}$ is **stationary in the wide or weak sense** if:

  $E[X_t] = E[X_{t+\tau}]$ and $COV(X_t, X_{t+\tau}) = C_X(\tau)$

  Constant expected value and **COV depends** only on the **delay**.

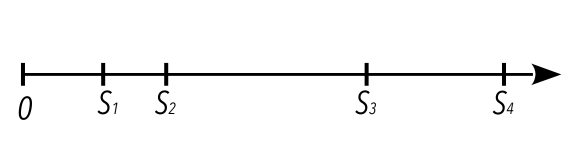
- $\{X\}_{t \in T}$ is **strictly stationary** if:
  $\forall t \in T$ the **rvs** X_t have the **same probability distrib.**

  Time passes but the **process distribution** remains the **same**.

Arrival processes

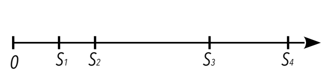
Arrival processes

- Useful to **model sequences of incidents**
  Departures, arrivals, ...
- **Incidents or arrivals** usually referred as **events (not accurate)**
- Process **starts at time 0**
- Multiple arrivals **cannot** happen simultaneously

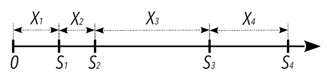


Arrival processes | Representations

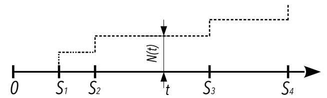
Arrival times : Joint distribution of $S_1, S_2, \ldots$



Inter-arrival times : Joint distribution of $X_1, X_2, \ldots$

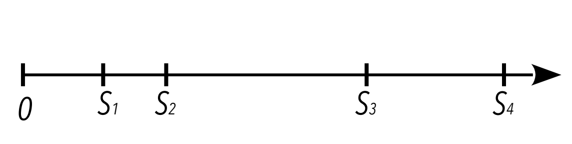


Counting rv: $\{N(t), \forall t > 0\}$



Arrival processes | Arrival times

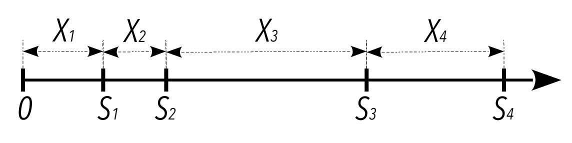
- **Sequence** of **increasing random variables** (rv)
  $0 < S_1 < S_2 < \dots$
- S_i : **arrival epoch** or **arrival time**
  Times when the phenomenon occurs.
- **Bulk arrivals**: associate a positive int. to each S_i.
- Example of event: $\{S_i \leq t\}$
  (the i-th arrival occurred at most at time t).



Arrival processes | Inter-arrival times

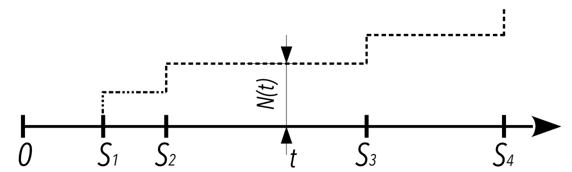
Sequence of inter-arrival times $X_1, X_2, \ldots$:

- $X_1 = S_1, \quad \text{and} \quad \forall i > 1 : X_i = S_{i+1} - S_i$
- $S_n = \sum_{i=1}^{n} X_i$
- X_i: positive rv
- $X_1, X_2, \ldots$ are usually iid



$N(t)$: **rv** denoting the **nb. of arrivals** in $]0, t]$, $\forall t > 0$.

- **Counting process** $\{N(t); t > 0\}$
- **By definition** $N(0) = 0$
- $\{S_n \leq t\} = \{N(t) \geq n\}$ (or $\{S_n > t\} = \{N(t) < n\}$)
    - $\{S_n \leq t\}$: The n-th arrival occured before time t.
    - $\{N(t) \geq n\}$: At time t at least n arrival occured.



Bernoulli process

Bernoulli Trial

- **Outcomes**: $\Omega = \{"failure", "success"\}$
- **rv** $X : \Omega \rightarrow \{0, 1\}$
  $X("sucess") = 1$ and $X("failure") = 0$.
- $P_X(1) = p$, $P_X(0) = q$ and $p + q = 1$
- e.g., experiment of **tossing a coin**.

Bernoulli process

Definition - arrival times:

- $\{X_i\}_{i \in N}; \quad N \subseteq \mathbb{N}$.
- $\forall i \in N, X_i$ is a **Bernoulli trial**.
- $X_1, X_2, \ldots$ are **iid**.

Inter-arrival times $\rightarrow$?

Counting $\rightarrow$?

Bernoulli process

- Inter-arrival times:
  - Number of trials needed to get next success.
  - Geometric distribution $p_X(j) = p \times (1-p)^{(j-1)}$.
- Counting:
  - Number of success in n first trials.
  - Binomial distribution $\mathcal{B}(n, p)$.

Poisson process

Poisson process

- Renewal process:
  - Arrival process
  - iid rvs inter-arrivals $X_1, X_2, \ldots$
- Poisson process:
  - Renewal process
  - s.t. $\forall i, X_i \sim$ exponential distribution
- Arrivals: Independent and not simultaneous.

Exponential distribution

- Continuous probability distribution.
- Parameter (rate): $\lambda \in \mathbb{R}_+^*$
- Support: $x \in \mathbb{R}_+$
- Mean: $\beta = \lambda^{-1}$
- Variance: $\beta^2 = \lambda^{-2}$

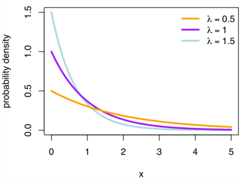


Figure 1: Prob. Density Func.: $f_X(x;\lambda) = \lambda e^{-\lambda x}$

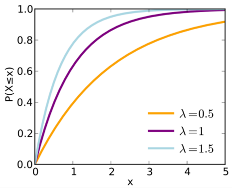


Figure 2: Cumul. Distrib. Func.: $F_X(x;\lambda) = P(X \leq x) = 1 - e^{-\lambda x}$

Memoryless property

Memoryless rv X if :

- $P(X > 0) = 1$
- $\forall \tau \geq 0$ and $t \geq 0$: $P(X > t + \tau) = P(X > \tau) \times P(X > t)$

True for **exponential rv** X of rate λ, $P(X > x) = e^{-\lambda x}$:

$P(X > x + t) = e^{-\lambda(x+t)} = e^{-\lambda x} \times e^{-\lambda t} = P(X > x) \cdot P(X > t)$.

Interpretation for waiting time:

$$P(X > t + \tau | X > t) = P(X > \tau)$$

The **remaining waiting time** has **no memory**.

Stationary increment property

Counting process $\{N(t), t > 0\}$ has the **stationary increment property** if:

$\forall (t, t') \in {\mathbb{R}_+^*}^2$ s.t. $0 < t < t'$ then:

$N(t') - N(t)$ and $N(t' - t)$ **have the same distribution**.

Independent increment property

Counting process $\{N(t), t > 0\}$ has the **independent increment property** if:

$\forall k \in \mathbb{N}^*$ s.t. $0 < t_1 < t_2 < \cdots < t_k$ then:

the k-tuple of rvs $N(t_1), N(t_2 - t_1), \ldots, N(t_k - t_{k-1})$ are **statistically independent**.

Theorem

Poisson processes have both:

- Stationary increment property
- Independent increment property

Probability density of S_n

- **Recall:** $S_n = \sum_{i=1}^{n} X_i$, sum of n iid rv
- $\forall x \geq 0$, each rv has a density funct. $f_X(x) = \lambda e^{(-\lambda x)}$
- Density of the sum of two independent rv $\rightarrow$ convolution of their densities.
- Prob. density of $S_n \sim$ **Erlang density**: $f_{S_n}(t) = \frac{\lambda^t t^{n-1} e^{-\lambda t}}{(n-1)!}$

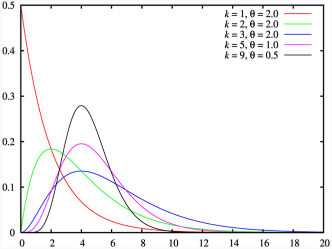


Figure 3: Erlang density function

Counting process probability mass function

$\{N(t)\}_{t>0}$ follows the well-known Poisson distribution.

$$P_{N(t)}(k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

- $\Omega = \{N(t) = 1, N(t) = 2, \ldots, N(t) = k\}$
- Poisson process rate: λ
- Mean: $\lambda \times t$
- Variance: $\lambda \times t$
- Proof based on $\{N(t) \geq n\} = \{S_n \leq t\}$

Combination of Poisson processes

- $N_1(t)_{t>0}$ and $N_2(t)_{t>0}$:
    - **Independent Poisson counting processes**
    - With **rates** λ_1 and λ_2 respectively.
- $N(t) = N_1(t) + N_2(t)$
- $\{N(t)\}_{t>0}$:
  Poisson counting process with **rate** $\lambda = \lambda_1 + \lambda_2$

Proof:?

Combination of Poisson processes

- $N_1(t)_{t>0}$ and $N_2(t)_{t>0}$:
  - Independent Poisson counting processes
  - With **rates** λ_1 and λ_2 respectively.
- $N(t) = N_1(t) + N_2(t)$
- $\{N(t)\}_{t>0}$:
  Poisson counting process with **rate** $\lambda = \lambda_1 + \lambda_2$

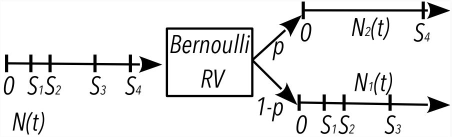
Proof:

- X_1 first inter-arrival for the joint process
- $X_1^{(1)}$ first inter-arrival for the first process
- $X_1^{(2)}$ first inter-arrival for the second process
- $P(X_1 > t) = P(X_1^{(1)} > t) \times P(X_1^{(2)} > t) = e^{-\lambda_1 t - \lambda_2 t} = e^{-\lambda t}$
- The memoryless property leads to the same conclusion for next arrivals.

Subdivision of Poisson process

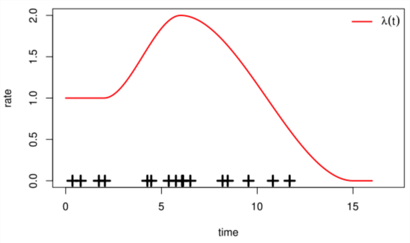
Break $\{N(t)\}_{t>0}$ Poisson counting process in two processes $\{N_1(t)\}_{t>0}$ and $\{N_2(t)\}_{t>0}$.

- $\forall i$ **label arrival** S_i as type **1** or **2** randomly.
- The **labeling** follows a **Bernoulli rv** X_n s.t. $p_{X_n}(1) = p$ and $p_{X_n}(2) = 1 - p$
- Resulting processes are **Poisson** with **rates**: $\lambda_1 = \lambda p$ and $\lambda_1 = \lambda(1 - p)$
- The two processes are **independent**.



Inhomogeneous Poisson Process

Location-dependent Poisson parameter $\lambda(t)$



Counting Probability:

$$P_{N_{[t_a,t_b[}}(k) = \frac{\Lambda(t_a,t_b)^k}{k!} e^{-\Lambda(t_a,t_b)}, \quad \Lambda(t_a,t_b) = \int_{t_a}^{t_b} \lambda(t)dt$$