

# Stochastic Process

## Markov, Moran and Branching

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# Markov Chains

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# Integer-time stochastic process

Stochastic process  $\{X_m\}_{m \in N}$  :

- Defined at **integer times**  $m \in N$ ,  $N \subseteq \mathbb{N}^*$
- The **rv**  $X_m$  is called **state at time**  $m$ .
- $X_m \in S$ , the **state space**.
  - $S = \mathbb{N} \rightarrow$  **Countable infinite set**
  - $S = \{1, \dots, N\} \rightarrow$  **Countable finite set**

Difference with counting process:

$\{N(t)\}_{t \in T} \in \mathbb{N}$  **but**  $t \in \mathbb{R}_+^*$ .

# Discrete-Time Markov Chain (MC)

Integer-time rand. process satisfying the **Markov property**:

Prob. of the **next state** only depends on the **current** one.

Integer-time rand. process  $\{X_1, X_2, \dots\}$  is a **MC** if  $\forall m > 1$ :

$$P(X_{m+1} = x_{m+1} | X_1 = x_1, \dots, X_m = x_m) = P(X_{m+1} = x_{m+1} | X_m = x_m)$$

i.e., the **future** state is **independent** from the **past** states **given** the **present** state.

$$X_{m+1} | X_m \perp\!\!\!\perp X_1, X_2, \dots, X_{m-1}$$

# Markov Chain of Order $n$

- Also known as **Markov chain with memory**
- The **next** state **depends** only on its **previous**  $n$  states.

MC of order  $n$ , if for  $n < m$ :

$$P(X_m = x_m | X_1 = x_1, \dots, X_{m-1} = x_{m-1}) = \\ P(X_m = x_m | X_{m-n} = x_{m-n}, \dots, X_{m-1} = x_{m-1})$$

Chain  $\{Y_m\}_{m \in \mathbb{N}}$ , s.t.  $\forall m > n, Y_m = (X_m, X_{m-1}, \dots, X_{m-n+1})$

has the **classical Markov property**

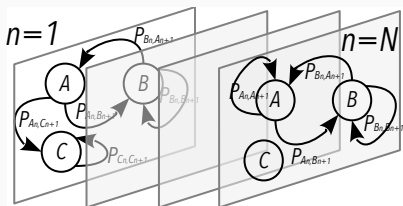
# Time-Homogeneous Markov Chain (HMC)

Also called **Stationary/Homogeneous Markov Chain**.

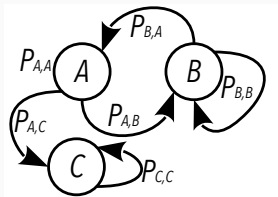
Prob. of state transition is independent of  $m$ , i.e.,  $\forall m$ :

$$P(X_{m+1} = x | X_m = y) = P(X_m = x | X_{m-1} = y)$$

# Markov Chain Representations



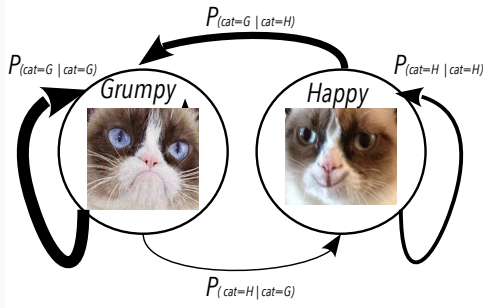
**Non-Homogeneous MC:**  
Sequence of digraphs



**Homogeneous MC:**  
Single digraph.

- **Nodes:** States
- **Edges:** Transition probabilities
- If  $P(X_{m+1} = j | X_m = i) = 0$ : arc  $\langle i, j \rangle$  omitted.

# Transition Matrix | HMC



$$M = \begin{bmatrix} & G & H \\ G & 0.9 & 0.1 \\ H & 0.7 & 0.3 \end{bmatrix}$$

- $\forall i, j, \quad M_{i,j} \geq 0$
- $\sum_j M_{i,j} = 1$

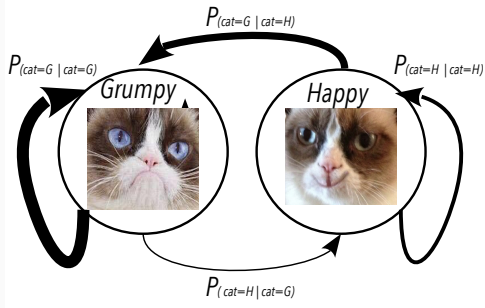
$M_{i,j} \rightarrow$  Prob. to go from state  $i$  to state  $j$

Example:

$$P(cat_{n+1} = G | cat_n = H) = ?; \quad P(cat_{n+1} = G | cat_n = G) = ?$$

$$P(cat_{n+1} = H | cat_n = G) = ?; \quad P(cat_{n+1} = H | cat_n = H) = ?$$

# Transition Matrix | HMC



$$M = \begin{bmatrix} & G & H \\ G & 0.9 & 0.1 \\ H & 0.7 & 0.3 \end{bmatrix}$$

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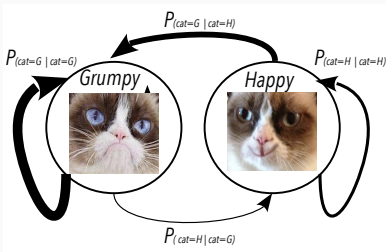
$M_{i,j} \rightarrow$  Prob. to go from state  $i$  to state  $j$

Example:

$$P(cat_{n+1} = G | cat_n = H) = .7; \quad P(cat_{n+1} = G | cat_n = G) = .9$$

$$P(cat_{n+1} = H | cat_n = G) = .1; \quad P(cat_{n+1} = H | cat_n = H) = .3$$

# Transitions | HMC



$$M = \begin{bmatrix} & G & H \\ G & 0.9 & 0.1 \\ H & 0.7 & 0.3 \end{bmatrix}$$

$\Pi^{(n)}$  → Distribution at step  $n$

$$\Pi^{(n+1)} = \Pi^{(n)} \cdot M$$

Example:

$$P(cat_{n+1} = H) = P(cat_n = H)M_{H,H} + P(cat_n = G)M_{G,H}$$

$$P(cat_{n+1} = G) = P(cat_n = G)M_{G,G} + P(cat_n = H)M_{H,G}$$

# $n$ -Step Transitions, Chapman-Kolmogorov Equation

Probability to go from state  $i$  to state  $j$  in  $n$  steps:

$$p_{i \rightarrow j}^{(n)} = P(X_n = j | X_0 = i)$$

If  $p_{i \rightarrow r}^{(k)} > 0$  and  $p_{r \rightarrow j}^{(n-k)} > 0$  then  $p_{i \rightarrow j}^{(n)} > 0$

Chapman-Kolmogorov equation:  $\forall k$  s.t.,  $0 \leq k \leq n$ :

$$p_{i \rightarrow j}^{(n)} = \sum_{r \in S} p_{i \rightarrow r}^{(k)} \cdot p_{r \rightarrow j}^{(n-k)}$$

For **HMC** (time independent):

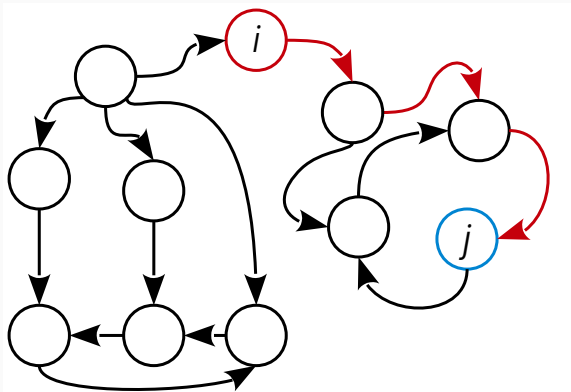
$$p_{i \rightarrow j}^{(n)} = P(X_{k+n} = j | X_k = i), \quad \forall k \text{ s.t., } 0 \leq k \leq n:$$

$$\Pi^{(n)} = \Pi^{(k)} \cdot M^{n-k}$$

# Graphic Representation | Accessibility

State  $j$  is **accessible** from state  $i$  (Notation:  $i \rightarrow j$ ) if:

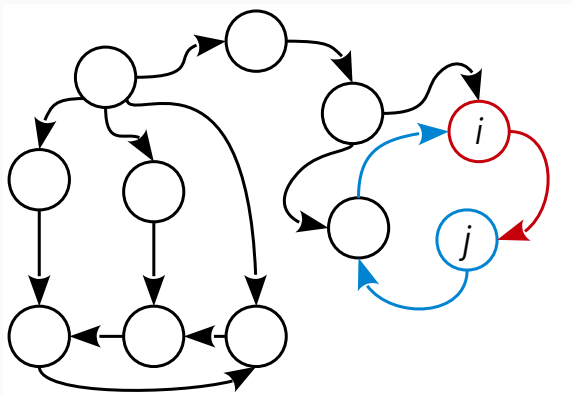
$$\exists n \geq 0, \quad p_{i \rightarrow j}^{(n)} > 0$$



# Graphic Representation | Communication

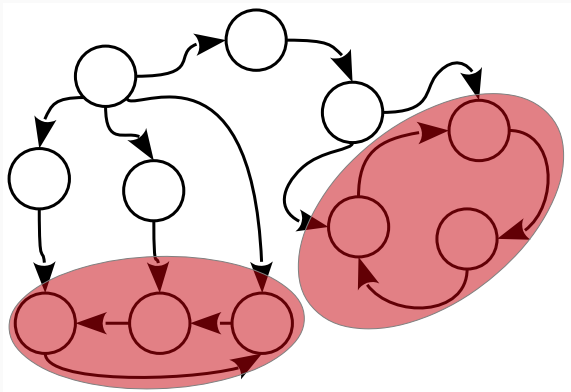
State  $j$  communicates with state  $i$  (Notation:  $i \rightarrow j$ ) if:

$$i \rightarrow j \text{ and } j \rightarrow i$$



# Graphic Representation | Communication Class

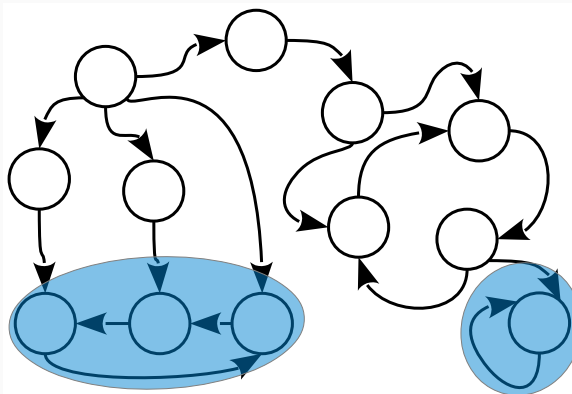
Maximal set of communicating states.



# Graphic Representation | Closed Communication Class

Communicating states s.t.  $\text{Prob.}(\text{leaving the class}) = 0$

i.e., no outgoing arrows

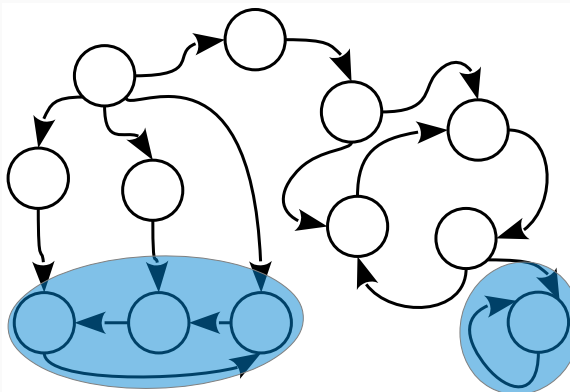


# Graphic Representation | Essential/Final State

State  $i$  is **essential/final** if:

$$\forall j \text{ s.t. } i \rightarrow j \text{ then } j \rightarrow i$$

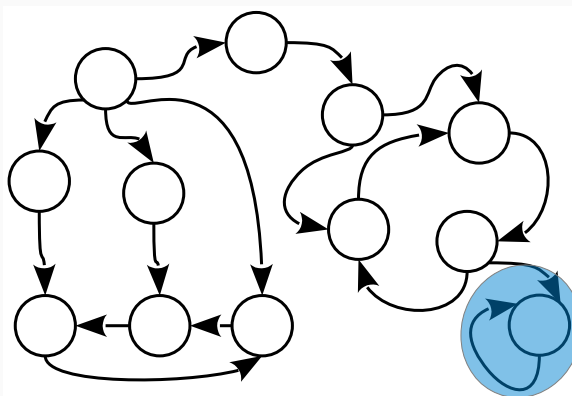
State  $i \in$  **closed communicating class**.



# Graphic Representation | Absorbing State

State  $i$  is absorbing if:

$$\forall n \geq 0 \quad P(X_{n+1} = i | X_n = i) = 1$$

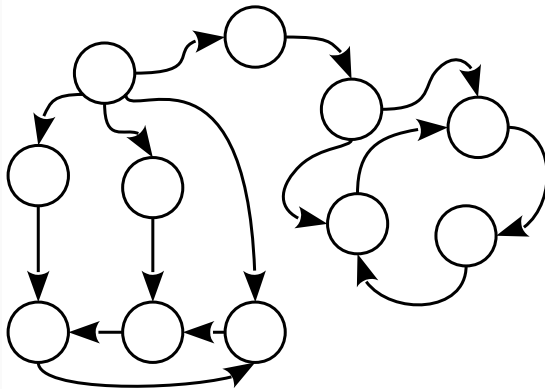


# Graphic Representation | Reducibility

Irreducible Markov Chain:

State space is a single communicating class

**Example:** Add edges to make an Irreducible Markov Chain



# Graphic Representation | Transience

Let  $T_i$  be a **rv** denoting the **first return time to state  $i$** .

$$T_i = \min\{n \geq 1 \quad \text{s.t.} \quad X_n = i | X_0 = i\}$$

- State  $i$  is **transient** if:

$$P(T_i = \infty) > 0 \quad \text{and} \quad P(T_i < \infty) < 1$$

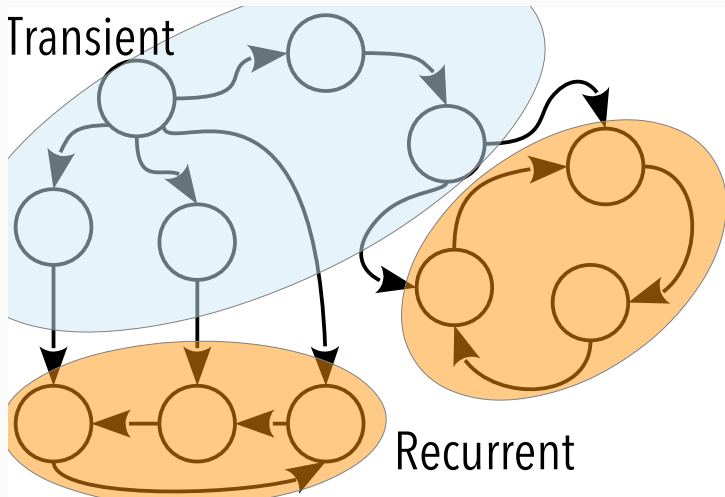
i.e., we are **not sure to come back**

- State  $i$  is **recurrent** or **persistent** if:

$$P(T_i = \infty) = 0 \quad \text{and} \quad P(T_i < \infty) = 1$$

i.e., we are **sure to come back**

# Graphic Representation | Transience



# Transience (exercise)

Prove the following theorem:

For any finite MC, the states in a communication class are all transient or all recurrent.

## Graphic Representation | Transience (proof)

- Let  $i \in S$  be **transient**  
and  $i \in C$  **communication class**.
- Thus:  $\exists j \in S$  s.t.  $i \rightarrow j$  but  $j \not\rightarrow i$
- Let  $m \in C \implies m \leftrightarrow i$
- Since  $m \rightarrow i$  and  $i \rightarrow j$  then:  $m \rightarrow j$
- If  $m$  is recurrent then:  $j \rightarrow m$
- Since  $j \rightarrow m$  and  $m \rightarrow i$  then:  $j \rightarrow i$
- **Contradiction:** so  $m$  is transient.

## Mean recurrence time

$$E[T_i] = \sum_{n=1}^{\infty} n \cdot P(T_i = n)$$

- State  $i$  is **positive recurrent** if:  $E[T_i]$  is **finite**.
- State  $i$  is **null recurrent** if:  $E[T_i]$  is **infinite**.

# Number of visits

$$N_i = \#\{n \geq 1 \text{ s.t. } X_n = i | N_0 = i\}$$

- $P(N_i = k) = P(T_i < \infty)^k$
- If state  $i$  is **recurrent**:  $P(N_i \geq 1) = 1$
- If state  $i$  is **transient**:  $\lim_{k \rightarrow +\infty} P(N_i = k) = 0$

# Periodicity

State  $i$  has **period**  $k_i$  if return times are multiples of  $k$ :

$$k_i = \gcd\{n > 0 \quad \text{s.t.} \quad P(X_n = i | X_0 = i) > 0\}$$

gcd: greatest common divisor

- State  $i$  is **aperiodic** if:  $k_i = 1$
- **Same period** for all states of communicating class
- MC **aperiodic** if: every state is aperiodic
- **Irreducible** MC with 1 aperiodic state  $\implies$  aperiodic

e.g., Bipartite graph  $\rightarrow$  even period

# Ergodicity

- State  $i$  is **ergodic** if:
  - $i$  is **aperiodic**: period = 1
  - $i$  is **positive recurrent**:  $P(T_i < \infty) = 1$  and  $E[T_i] < \infty$
- MC is **ergodic** if:
  - Every state  $i \in S$  is **ergodic**.
  - $\Leftrightarrow$  **Irreducible aperiodic and recurrent MC**

# Ergodic Theorem | Steady-State

Theorem: Let  $X_t$  be a **ergodic MC** then:

$$\lim_{n \rightarrow +\infty} p(X_n = j) = \Pi_j^*$$

Corollaries:  $\Pi^*$  is an **Stationary distribution**:

- $0 \leq \Pi_i^* \leq 1, \quad \forall i \in S$
- $\sum_i \Pi_i^* = 1$
- $\Pi_j^* = \sum_{i \in S} \Pi_i^* p_{i \rightarrow j} \Leftrightarrow \Pi^* = \boxed{\Pi^* \cdot M}$   
 $\Pi^*$ : normalized left  $M$  eigenvector, with eigenvalue 1.
- $\Pi^*$  is unique:  $\Pi_i^* = \frac{C}{E[T_i]}$  with  $C > 0$  a constant
- Distribution converges to  $\Pi^*$  regardless of  $\Pi^{(0)}$ .

# Perron-Frobenius Theorem

Let  $A$  be a  $n \times n$  **positive matrix** or a **non-negative irreducible matrix**<sup>1</sup>, with **unitary period**, then:

$$\exists r \in \mathbb{R}_+^* \quad \text{and} \quad \exists v = (v_1, \dots, v_n) \quad \text{s.t.} \quad M \cdot v = r \cdot v$$

$r$ : **eigenvalue**<sup>2</sup> and  $v$ : **eigenvector**<sup>3</sup> of  $M$  s.t.:

- **Other eigenvalues**  $\lambda$  are s.t.,  $|\lambda| < r$ .
- $\forall i \quad v_i > 0$ , and  $\nexists$  **other positive eigenvectors**.

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<sup>1</sup>Adjacency matrix of a strongly connected graph

<sup>2</sup>Perron-Frobenius/leading/dominant eigenvalue

<sup>3</sup>Perron-Frobenius/leading/dominant eigenvector

# Perron-Frobenius Theorem

Consequence:

Finite irreducible HMC with transition matrix  $M$ .

$$\boxed{\exists! \Pi \quad \text{s.t.} \quad \Pi \cdot M = \Pi}$$

Only one Steady-State distribution vector.

$$\boxed{\Pi_n = \Pi_0 \cdot M^n \quad \text{and} \quad \lim_{n \rightarrow \infty} \Pi_n = \Pi \quad \text{Then} \quad \lim_{n \rightarrow \infty} \Pi_0 \cdot M^n = \Pi}$$

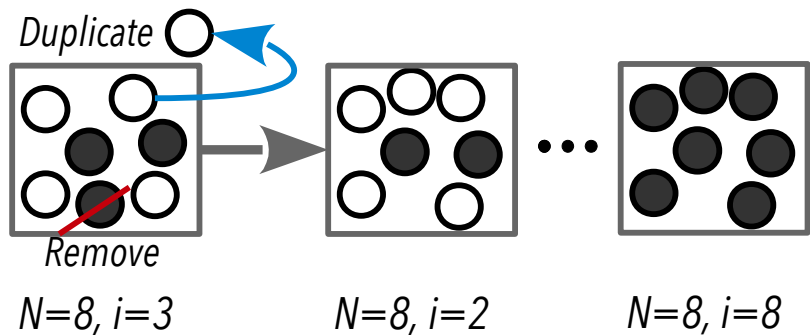
# Moran Process

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# Moran process

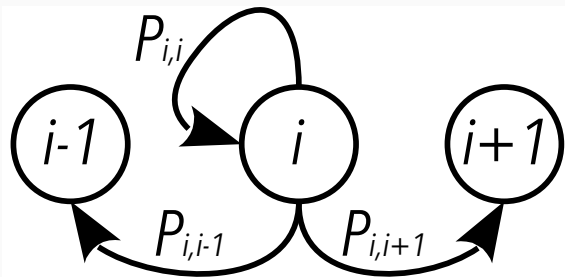
- Simple stochastic process used in biology
  - Model **finite populations** ( $N$  individuals)
  - **Variety-increasing effects** (e.g., mutations)
  - **variety-reducing effects** (e.g., drift, selection)
- Main characteristics
  - **Constant population size**  $N$
  - **Two populations**: Vector  $(i, N - i)$  with  $i \geq 0$ .
- At each **iteration**:
  - **Reproduce** 1 individual at **random**
  - **Kill** 1 individual at **random**  
(The **same** individual **can** be **chosen twice**)
  - If **only 1 type** of individuals  $\rightarrow$  **End**

# Moran process



# Moran process

Markov Process with state  $i$ , (i.e, nb. of individuals of type 1)



$$P_{i,i} + P_{i,i-1} + P_{i,i+1} = 1$$

Absorbing states: 0 and  $N$ .

# Fixation probability | Birth death process

$x_i$  : Probability of reaching state  $N$  from state  $i$ .

$$x_i = \begin{cases} 0 & \text{if } i = 0 \\ p_{i,i-1}x_{i-1} + p_{i,i+1}x_{i+1} + p_{i,i}x_i & \text{if } 0 < i < N \\ 1 & \text{if } i = N \end{cases}$$

**Theorem:** Fixation probabilities

$$x_i = \frac{1 + \sum_{j=1}^{i-1} \prod_{k=1}^j \gamma_k}{1 + \sum_{j=1}^{N-1} \prod_{k=1}^j \gamma_k}$$

With  $\gamma_k = \frac{p_{k,k-1}}{p_{k,k+1}}$

**Proof:?** Hint: Compute  $x_{i+1} - x_i$

# Proof

$$p_{i,i+1}x_{i+1} = x_i(1 - p_{i,i}) - p_{i,i-1}x_{i-1}$$

$$p_{i,i+1}x_{i+1} = p_{i,i-1}(x_i - x_{i-1}) + p_{i,i+1}x_i$$

$$x_{i+1} - x_i = \frac{p_{i,i-1}}{p_{i,i+1}}(x_i - x_{i-1}) = \gamma_i(x_i - x_{i-1})$$

$$x_{i+1} - x_i = \prod_{k=1}^i \gamma_k(x_1 - x_0) = \prod_{k=1}^i \gamma_k x_1.$$

Since  $x_0 = 0$ .

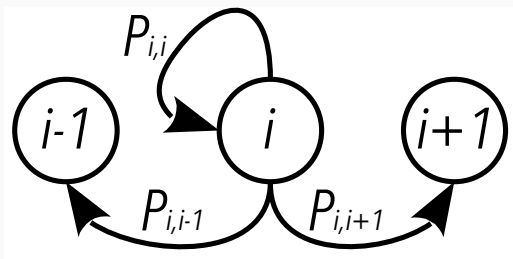
$$x_i = x_i - x_0 = \sum_{j=0}^{i-1} x_{j+1} - x_j = x_1(1 + \sum_{j=1}^{i-1} \prod_{k=1}^j \gamma_k)$$

$$\text{Since } x_N = 1: \quad x_1 = \frac{1}{1 + \sum_{j=1}^{N-1} \prod_{k=1}^j \gamma_k}$$

$$\text{Finally } x_i = \frac{1 + \sum_{j=1}^{i-1} \prod_{k=1}^j \gamma_k}{1 + \sum_{j=1}^{N-1} \prod_{k=1}^j \gamma_k}$$

# Neutral drift

Markov Process with state  $i$ , i.e. nb. of individuals of type 1):



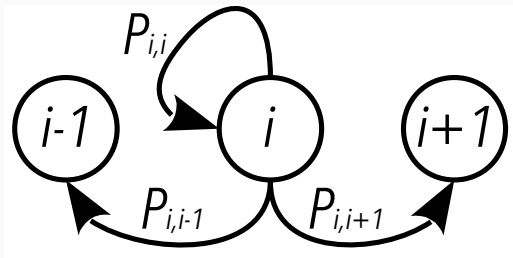
- $p_{i,i-1} = \frac{i(N-i)}{N^2}$
- $p_{i,i+1} = \frac{i(N-i)}{N^2}$
- $p_{i,i} = 1 - p_{i,i-1} - p_{i,i+1}$

Absorbing states: 0 and  $N$ .

$$p_{i,i+1} = p_{i,i-1} \implies \gamma_i = 1$$

# Selection

Markov Process with state  $i$ , i.e. nb. of individuals of type 1  
Fitness:  $f$  (type 1) and  $g$  (type 2).



- $p_{i,i-1} = \frac{g \cdot (N-i)}{f \cdot i + g \cdot (N-i)} \frac{i}{N}$
- $p_{i,i+1} = \frac{f \cdot i}{f \cdot i + g \cdot (N-i)} \frac{N-i}{N}$
- $p_{i,i} = 1 - p_{i,i-1} - p_{i,i+1}$

Absorbing states: 0 and  $N$ .

# Selection | Fixation Probability

- **Fixation:** Prob. to take over the **entire population**
- $\gamma_i = \frac{P_{i,i-1}}{P_{i,i+1}} = \frac{g_i}{f_i} = \frac{1}{r}$

$$X_i = \frac{1 - r^{-i}}{1 - r^{-N}}$$

## Selection | Evolution Rate

- Population 1 → **mutants**
- Only one mutant  $i = 1$
- **Fixation**: Prob. mutants take over the **entire population**

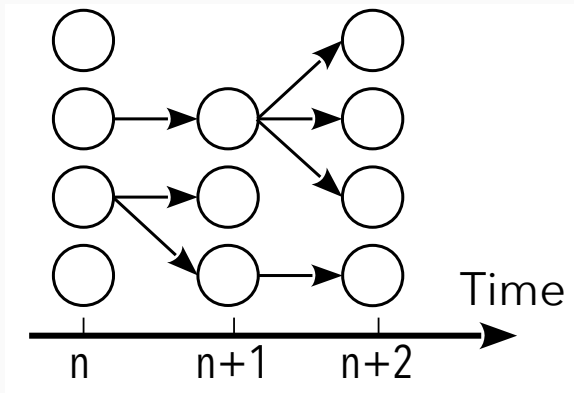
$$\rho = x_1 = \frac{1 - r^{-1}}{1 - r^{-N}}$$

# Branching Process

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# Branching Process

Population of individuals producing, each generation  $n$ , a random number of children.

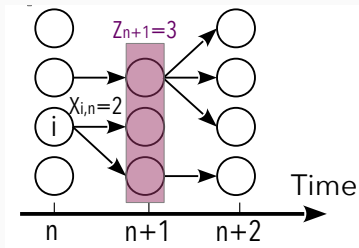


# Formal Definition | Galton–Watson process

- Individual lifespan = 1.
- **nb. children** of indiv.  $i$  at  $n$ .  
 $X_{i,n} \in \{0, 1, 2, \dots\}$
- $P(X_{i,n} = k) = p_k$
- $X_{i,n}$  are iid rv.
- **State/size of generation:**  
 $Z_{n+1} = \sum_{i=1}^{Z_n} X_{i,n}; \quad Z_0 = 1$

- **Branching Process:**

$$\{Z_n\}_{n \in \mathbb{N}}$$



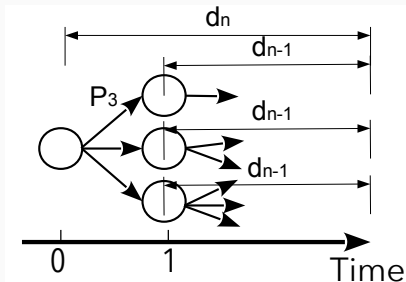
# Extinction problem

Extinction probability in  $n$  generations:

$$d_n = p_0 + p_1 d_{n-1} + p_2 d_{n-1}^2 + p_3 d_{n-1}^3 + \dots = h(d_{n-1})$$

Ultimate Extinction probability:

$$d = \lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} P(Z_n = 0) \quad \text{and thus: } \boxed{d = h(d)}$$

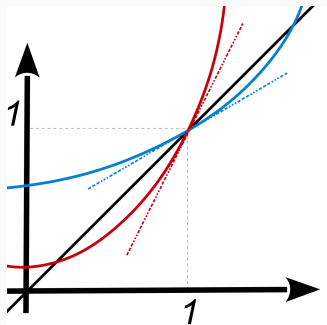


# Extinction problem

$$\begin{cases} h'(d) = p_1 + 2p_2d + 3p_3d^2 + \dots \geq 0 & \implies h(d) \text{ is increasing} \\ h''(d) = 2p_2 + 6p_3d + 12p_4d^2 \dots \geq 0 & \implies h(d) \text{ is convex} \end{cases}$$

$d = h(d)$  and  $d = 1$  is a solution  $\rightarrow \exists 3$  cases:

- $\exists$  Another intersect  $< 1$   
 $\rightarrow d < 1$  or  $d = 1$ .
- $d = 1$  is the only intersect  
 $\rightarrow d = 1$ .
- $\exists$  Another intersect  $> 1$   
 $\rightarrow d = 1$  (since  $0 \geq d \geq 1$ ).



# Extinction problem

$$h'(1) = p_1 + 2p_2 + 3p_3 + \dots = \mathbb{E}(X_{i,n})$$

$$\begin{cases} \mathbb{E}(X_{i,n}) \leq 1 & \implies d = 1 \\ \mathbb{E}(X_{i,n}) > 1 & \implies d < 1 \text{ or } d = 1 \end{cases}$$

