

Stochastic Process

Reminders of Probability Theory

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Questions

- **Definitions:** Sample space, event, complement event, prob. measure, random variable, cumulative distribution function, probability density function, probability mass function, expected value, variance, indicator, relative/absolute frequency, iid, CLT.
- Choose 3 **core components** of probabilities: Sample space, event, prob. measure, std, average, sampling, frequencies.
- **Recall:** Axioms of events, axioms of probabilities, conditional probabilities formula, Bayes formula, independence between 2 events.

Definitions

- Sample space (Ω): ?
- Event $A \in \mathcal{F}$: ?
- Complement event: ?
- Prob. measure $P(\cdot)$: ?
- Prob. of an event: ?
- Random Variable: ?
- Core components: ?

Exemplify each concept considering a six-sided rolling die experiment.

Definitions

- **Sample space** (Ω): arbitrary set of *sample points* or *outcomes*.
- **Event** $A \in \mathcal{F}$: subset of the same space $A \subseteq \Omega$
- **Complement event** $\bar{A} = \Omega \setminus A$
- **Prob. measure** $P(\cdot)$: Assignment of a non-negative number to each outcome
- **Prob. of an event**: Sum of the the prob. of its outcomes.
- **Random Variable**: Mapping function $X : \Omega \rightarrow \mathbb{R}$.
- **Core components**:
Probability space (Ω, \mathcal{F}, P)

Exemplify each concept considering a six-sided rolling die experiment.

Events axioms?

Probabilities axioms?

Axioms

Events axioms

- Ω is an event.
- \emptyset is an event.
- if A_1, A_2, \dots are events, then $\bigcup_n A_n$ and $\bigcap_n A_n$ are events.

Probabilities axioms

- $P(\Omega) = 1$.
- If A is an event $P(A) \geq 0$.
- If A_1, A_2, \dots are disjoint events then
$$P\left(\bigcup_n A_n\right) = \sum_n P(A_n).$$

Some Consequences

- $P(\emptyset) = ?$.
- $P(\bar{A}) = ?$.
- If $A \subseteq B$ then $P(A) \leq P(B)$.
- $P(\bigcup_n A_n) \leq \sum_n P(A_n)$.

Some Consequences

- $P(\emptyset) = 0$.
- $P(\bar{A}) = 1 - P(A)$.
- If $A \subseteq B$ then $P(A) \leq P(B)$.
- $P(\bigcup_n A_n) \leq \sum_n P(A_n)$.

Little test

"Linda is thirty-one years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in antinuclear demonstrations "

Kahneman, Daniel. *Thinking, Fast and Slow*.

Which sentence is more likely?

A) "Linda is a bank teller"

B) "Linda is a bank teller and is active in the feminist movement"

Definitions

- Conditional prob. of A , conditional to B :?
- Bayes formula: ?
- 2 events A and B are **independent** if ...

Definitions

- **Conditional prob.** of A, conditional to B:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- **Bayes formula:** $P(A|B) = \frac{P(B|A) \times P(A)}{P(B)}$

- 2 events A and B are **independent** if

$$P(A|B) = P(A) \text{ or } P(A \cap B) = P(A) \times P(B)$$

Little test

”Steve is very shy and withdrawn, invariably helpful but with little interest in people or in the world of reality. A meek and tidy soul, he has a need for order and structure and stre, and a passion for detail. Is Steve more likely to be a librarian or a farmer? ”

Kahneman, Daniel. *Thinking, Fast and Slow*

Definitions

- (Cumulative) Distribution function (CDF): ?
- Probability Density Function (PDF): ?
- Probability Mass Function (PMF): ?

Definitions

- **Cumulative Distribution function:** $F_X : \mathbb{R} \rightarrow [0, 1]$; s.t. $F_X(x) = P(X \leq x)$, non-decreasing from 0 to 1.
- **Probability Density Function:** $f_X(x) = \frac{\partial F(x)}{\partial x}$
(if F_X is differentiable).
- **Probability Mass Function:**
 $f_X : \mathbb{R} \rightarrow [0, 1]$; s.t. $f_X(x) = P_X(x)$

Definitions

- Expected value (discrete): ?
- Variance: ?

Definitions

- Expected value (discrete): $E[X] = \bar{X} = \sum_x xP_X(x)$
- Variance: $\sigma_X^2 = E[(X - \bar{X})^2] = E[X^2] - \bar{X}^2$
- $E[aX + bY] = ?$
- $\text{VAR}(aX + bY) = \sigma_{aX+bY}^2 = ?$

Definitions

- **Expected value (discrete):** $E[X] = \bar{X} = \sum_x xP_X(x)$
- **Variance:** $\sigma_X^2 = E[(X - \bar{X})^2] = E[X^2] - \bar{X}^2$
- $E[aX + bY] = aE[X] + bE[Y]$
- $VAR(aX + bY) = a^2VAR(X) + b^2VAR(Y) + 2abCOV(X, Y)$

Indicator:?

Definitions

Indicator: $\mathbb{I}_A : \Omega \rightarrow \{0, 1\}$, s.t. for $\omega \in \Omega$ and an event A , $\mathbb{I}_A(\omega) = 1$ if $\omega \in A$, and 0 otherwise.

- $P_{\mathbb{I}_A}(1) = ?$
- $P_{\mathbb{I}_A}(0) = ?$
- $E[\mathbb{I}_A] = ?$
- $\sigma_{\mathbb{I}_A}^2 = ?$

Definitions

Indicator: $\mathbb{I}_A : \Omega \rightarrow \{0, 1\}$, s.t. for $\omega \in \Omega$ and an event A , $\mathbb{I}_A(\omega) = 1$ if $\omega \in A$, and 0 otherwise.

- $P_{\mathbb{I}_A}(1) = P(A)$
- $P_{\mathbb{I}_A}(0) = 1 - P(A)$
- $E[\mathbb{I}_A] = P(A)$
- $\sigma_{\mathbb{I}_A}^2 = P(A)(1 - P(A))$

Definitions

- Frequency:
 - Absolute frequency ?
 - Relative frequency ?
- Independent and Identically Distributed (IID)
random variables: ?

Definitions

- **Frequency:**
 - **Absolute frequency** n_A number of times the event A occurred in an experiment.
 - **Relative frequency** $\frac{n_A}{\sum_A n_A}$
 - Intuition to grasp probabilities.
- **Independent and Identically Distributed (IID)** random variables:

Sequence of random variables s.t.

- Each variable has the **same probability function**.
- Variables are **mutually independent**.

Central Limit Theorem (CLT): ?

Central Limit Theorem (CLT)

- X_1, X_2, \dots sequence of iid random variables with expected value \bar{X} and variance σ_X^2 .
- $S_n = \sum_{i=0}^n X_n$.
- for large n , $\frac{S_n - n\bar{X}}{\sigma_X \sqrt{n}} \sim \mathcal{N}(0, 1)$.

Definitions: Gaussian Distribution

- Parameters: $\mu \in \mathbb{R}, \sigma \in \mathbb{R}_+^*$
- Mean μ
- Variance σ^2

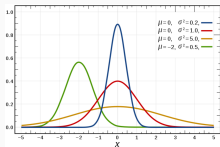


Figure 1: Probability Density Function:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

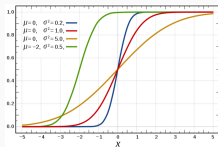


Figure 2: Cumulative Distribution Function:

$$F_X(x) = \frac{1}{2} \left(1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right)$$